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Spectral Resolution of a Special Type of Operator, Called λ-Jection of Third Order

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ABSTRACT

In this paper, I consider an operator called λ-jection or a λ-jection of third order and obtain its spectral resolution.

Keywords : λ-Jection, Projection, Spectrum, Spectral Resolution

I. INTRODUCTION

Dr. P Chandra defined a trijection operator in his Ph.D. thesis titled "Investigation into the theory of operators and linear spaces".[1] In Dunford, N. and Schwartz, J. [2], p.37 and Rudin [3], p.126 a projection operator E has been defined as $E^2 = E$. E is a trijection operator if $E^3 = E$. To generalise it, I have defined E to be a λ-jection (of third order) [5] if

$$E^3 + \lambda E^2 = (1 + \lambda)E, \qquad \lambda \textit{ being a scalar}.$$

In case $\lambda = 0$, we have a trijection. In case E is a projection, it is also a λ-jection.

II. Definition

Let H be a Hilbert space and E an operator on H. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be eigen values of E and $M_1, M_2, \ldots, M_m$ be their corresponding eigen spaces. Let $P_1, P_2, \ldots, P_m$ be the projections on these eigen spaces. Then according to definition of spectral theorem in Simmons [4], p 279-290, the following statements are all equivalent to one another.

1. The $M_i's$ are pairwise orthogonal and span H.
2. The $P_i's$ are pairwise orthogonal, $I = \sum_{i=1}^{m} P_i$ and $E = \sum_{i=1}^{m} \lambda_i P_i$

3. E is normal.
4. Then the set of eigen values of E is called its spectrum and is denoted by $\sigma(E)$. Also

$$E = \lambda_1 P_1 + \lambda_2 P_2 + \ldots. + \lambda_m P_m$$

Expression for E given above is called the spectral resolution of E.

III. Main Result

Theorem 1

Let E be a λ-jection on a Hilbert space H. Assume $\lambda \neq -1 \; or -2$. Then E can be expressed as a linear combination of two pairwise orthogonal projections.

Proof :-

E is a λ-jection if

$E^3 + \lambda E^2 = (1+\lambda)E$

Let, $\lambda + 1 = \mu \; or \; \lambda = \mu - 1$, then

$E^3 + (\mu - 1)E^2 = \mu E$

$\Rightarrow E^3 - E^2 = \mu E - \mu E^2 = \mu(E - E^2)$

$\Rightarrow E^3 = \mu(E - E^2) + E^2 = \mu E - (\mu - 1)E^2$ ------------------------------------ (1)

Applying E to both sides,

$E^4 = \mu E^2 - (\mu - 1)E^3 = \mu E^2 - (\mu - 1)\{\mu E - (\mu - 1)E^2\}$

$= \mu E^2 - \mu(\mu - 1)E + (\mu - 1)^2 E^2$

$= -(\mu^2 - \mu)E + \{\mu + (\mu - 1)^2\}E^2$

$= (\mu - \mu^2)E + (1 - \mu + \mu^2)E^2$ -- (2)

Let E can be expressed as

$E = aP_1 + bP_2$ -- (3)

Where a and b are scalars, P_1, P_2 pairwise orthogonal projections i.e.-

$P_1^2 = P_1, P_2^2 = P_2, P_1P_2 = 0$

Hence squaring E,

$E^2 = a^2P_1 + b^2P_2$ -- (4)

Solving (3) and (4) for P_1 and P_2 in terms of E and E^2,

$P_1 = \frac{E^2 - bE}{a(a-b)}, P_2 = \frac{aE - E^2}{b(a-b)}$ -- (5)

Since $P_1P_2 = 0$,

$$\frac{E^2 - bE}{a(a-b)} * \frac{aE - E^2}{b(a-b)} = 0$$

$\Rightarrow (E^2 - bE) * (aE - E^2) = 0$

$\Rightarrow aE^3 - E^4 - abE^2 + bE^3 = 0$

$\Rightarrow (a+b)E^3 - E^4 - abE^2 = 0$

$\Rightarrow (a+b)[\mu E - (\mu - 1)E^2] - [(\mu - \mu^2)E + (1 - \mu + \mu^2)E^2] - abE^2 = 0$ using (1) and (2)

Equating coefficients of E on both sides,

$\mu(a+b) - (\mu - \mu^2) = 0$

Since $\mu \neq 0 \; as \; \lambda \neq -1$,

$(a+b) - (1-\mu) = 0 \;\Rightarrow a + b = 1 - \mu$ --------------------------------- (6)

Equating coefficients of E^2 on both sides,

$-(a+b)(\mu-1)-(1-\mu+\mu^2)-ab=0$
Using (6),
$(1-\mu)^2-(1-\mu+\mu^2)=ab$
$\Rightarrow ab=1-2\mu+\mu^2-1+\mu-\mu^2=-\mu$
Hence $(a-b)^2=(a+b)^2-4ab=(1-\mu)^2+4\mu=(1+\mu)^2$
So $a-b=\pm(1+\mu)$
Let $a-b=1+\mu. Also\ a+b=1-\mu$
Hence $a=1, b=-\mu$
Therefore $P_1=\frac{E^2-bE}{a(a-b)}=\frac{E^2+\mu E}{1+\mu}\quad (\mu\neq -1\ as\ \lambda\neq -2)$
And $P_2=\frac{aE-E^2}{b(a-b)}=\frac{E-E^2}{-\mu(1+\mu)}=\frac{E^2-E}{\mu(1+\mu)}$
If we consider $a-b=-(1+\mu)$, we get same values of P_1 and P_2
Hence $E=P_1-\mu P_2$
Thus we have expressed E as a linear combination of two pairwise orthogonal projections.

Theorem 2
Let E be a λ-jection on Hilbert space H. There are three pairwise orthogonal projections P_1,P_2,P_3 such that
$E=\lambda_1 P_1+\lambda_2 P_2+\lambda_3 P_3$
Where $\lambda_1, \lambda_2\ \lambda_3$ are scalars and
$I=P_1+P_2+P_3$
Proof:-
We have seen in theorem 1 that
$E=P_1-\mu P_2$
Where P_1,P_2 are orthogonal projections.
Let $Q=P_1+P_2$
Then $Q^2=P_1^2+P_2^2+2P_1P_2=P_1+P_2$
So Q is also a projection. Hence $I-Q$ is also a projection.
Let $P_3=I-Q=I-P_1-P_2$
Then P_3 is a projection such that
$P_1P_3=P_1(I-P_1-P_2)=P_1-P_1^2-P_1P_2=0$
$P_2P_3=P_2(I-P_1-P_2)=P_2-P_2^2-P_1P_2=0$
Thus P_1, P_2, P_3 are pairwise orthogonal.
Moreover, $P_1+P_2+P_3=P_1+P_2+I-P_1-P_2=I$
Choose $\lambda_1=1, \lambda_2=-\mu, and\ \lambda_3=0$
Then $E=P_1-\mu P_2=P_1-\mu P_2+0*P_3=\lambda_1P_1+\lambda_2P_2+\lambda_3P_3$
Where $\lambda_1=1, \lambda_2=-\mu\ and\ \lambda_3=0$

Theorem 3
Range of projection P_1 denoted by R_{P_1} is given by
$R_{P_1}=\{z: Ez=z\}=M_1(say)$
Proof :
Let $z\in R_P$, then since P_1 is a projection, $P_1z=z$

Now $EP_1 = \frac{E(E^2+\mu E)}{\mu+1} = \frac{E^3+\mu E^2}{\mu+1} = \frac{\mu E+(1-\mu)E^2+\mu E^2}{\mu+1} = \frac{\mu E+E^2}{\mu+1} = P_1$

Hence $Ez = E(P_1 z) = EP_1 z = P_1 z = z$

So $z \in M_1$

Hence $R_{P_1} \subseteq M_1$ -- (7)

Conversely, let $z \in M_1$, i.e. $Ez = z$

Then $E^2 z = E(Ez) = Ez = z$

So $P_1 z = \frac{(E^2+\mu E)}{\mu+1} z = \frac{E^2 z+\mu E z}{\mu+1} = \frac{z+\mu z}{\mu+1} = \frac{(1+\mu)z}{\mu+1} = z$

Thus $z \in R_{P_1}$

So $M_1 \subseteq R_{P_1}$ -- (8)

From (7) and (8)

$R_{P_1} = M_1$

Theorem 4

We show that

$R_{P_2} = \{z: Ez = -\mu z\} = M_2(say)$

Proof:-

Let $z \in R_{P_2}$ then $P_2 z = z$

Now $EP_2 = \frac{E(E^2-E)}{\mu(\mu+1)} = \frac{E^3-E^2}{\mu(\mu+1)} = \frac{\mu(E-E^2)}{\mu(\mu+1)} = \frac{-\mu(E^2-E)}{\mu(\mu+1)} = -\mu P_2$

So $Ez = E(P_2 z) = EP_2 z = (-\mu P_2)z = -\mu z$

$\Rightarrow z \in M_2$

Hence $R_{P_2} \subseteq M_2$ -- (9)

Conversely, *let* $z \in M_2$, then $Ez = -\mu z$

Then $E^2 z = E(Ez) = E(-\mu z) = -\mu Ez = \mu^2 z$

Hence $P_2 z = \frac{(E^2-E)}{\mu(\mu+1)} z = \frac{\mu^2 z+\mu z}{\mu^2+\mu} = z$

So $z \in R_{P_2}$

Thus $M_2 \subseteq R_{P_2}$ -- (10)

From (9) and (10),

$R_{P_2} = M_2$

Theorem 5

We show that

$R_{P_3} = \{z: Ez = 0\} = M_3(say)$

Proof:-

We have

$P_3 = I - P_1 - P_2$

Let $z \in R_{P_3}$ *then* $P_3 z = z$

Now $EP_3 = E(I - P_1 - P_2) = E - EP_1 - EP_2$

$= E - P_1 + \mu P_2 = E - (P_1 - \mu P_2) = E - E = 0$

Using Theorems 1,3 and 4

So, $Ez = E(P_3z) = EP_3z = 0z = 0$
$\Rightarrow z \in M_3$
Hence $R_{P_3} \subseteq M_3$ --- (11)
Let $z \in M_3, then\ Ez = 0 \Rightarrow E^2z = 0$
Then

$$P_1z = \frac{(E^2 + \mu E)}{\mu + 1}z = \frac{E^2z + \mu Ez}{\mu + 1} = 0$$

$$P_2z = \frac{(E^2 - E)}{\mu(\mu + 1)}z = \frac{E^2z - Ez}{\mu(\mu + 1)} = 0$$

Hence $P_3z = [I - P_1 - P_2]z = z - 0 - 0 = z$
So $z \in R_{P_3}$
Hence $M_3 \subseteq R_{P_3}$ -- (12)
From (11) and (12)
$R_{P_3} = M_3$

Theorem 6

Let E be a λ-jection on a Hilbert space H. Let $\lambda_1 = 1, \lambda_2 = -\mu\ and\ \lambda_3 = 0$ be eigen values of E. M_1, M_2, M_3 be their corresponding eigen spaces. Let P_1, P_2, P_3 be projections on these eigen spaces where

$$P_1 = \frac{E^2 + (\lambda + 1)E}{\lambda + 2}, P_2 = \frac{E^2 - E}{(\lambda + 1)(\lambda + 2)}, P_3 = I - P_1 - P_2$$

Then $P_1 + P_2 + P_3 = I$
$P_i's$ are pairwise orthogonal and spectral resolution of E is given by (assuming $\lambda \neq -1\ or\ -2$)
$E = \lambda_1P_1 + \lambda_2P_2 + \lambda_3P_3$
And spectrum of E is given by
$\sigma(E) = \{1, -(\lambda + 1), 0\}$

Proof:-

Theorems 3,4,5 show that $\lambda_1 = 1, \lambda_2 = -\mu, \lambda_3 = 0$ are eigen values of E, M_1, M_2, M_3 are their corresponding eigen spaces and P_1, P_2, P_3 are pairwise orthogonal projections on these eigen spaces. Also due to theorem 2,
$E = \lambda_1P_1 + \lambda_2P_2 + \lambda_3P_3,$
$I = P_1 + P_2 + P_3$
Hence expression for E given above is the spectral resolution of E. Also the spectrum of E, since the eigen values of E are 1, $-\mu$, and 0 is given by
$\sigma(E) = \{1, -(\lambda + 1), 0\}$ as $\mu = \lambda + 1$

IV. References

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