Fixed Point Theorems in Fuzzy Metric Spaces

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ABSTRACT

In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).

Keywords : Fixed point, Fuzzy metric spaces, Fuzzy mapping.

I. INTRODUCTION

In 1965, the theory of fuzzy sets was investigated by Zadeh [2]. In 1981, Heilpern [3] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [4]. Therefore, several fixed point theorems for types of fuzzy contractive mappings have appeared (see, for instance [1,5–9]).

In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).

II. METHODS AND MATERIAL

Basic Preliminaries

The definitions and terminologies for further discussions are taken from Heilpern [3]. Let \((X, \alpha)\) be a metric linear space. A fuzzy set in \(X\) is a function with domain \(X\) and values in \([0, 1]\). If \(A\) is a fuzzy set and \(x \in X\), then the function-value \(A(x)\) is called the grade of membership of \(x\) in \(A\). The collection of all fuzzy sets in \(X\) is denoted by \(I(X)\).

Let \(A \in I(X)\) and \(\alpha \in [0, 1]\). The \(\alpha\)-level set of \(A\), denoted by \(A_{\alpha}\), is defined by

\[A_{\alpha} = \{x: A(x) \geq \alpha\}\]

whenever \(\bar{B}\) is the closure of set (non-fuzzy) \(B\).

Definition 2.1

A fuzzy set \(A\) in \(X\) is an approximate quantity iff its \(\alpha\)-level set is a nonempty compact convex subset (non-fuzzy) of \(X\) for each \(\alpha \in [0, 1]\) and \(\sup_{x \in X} A(x) = 1\). The set of all approximate quantities, denoted by \(W(X)\), is a subcollection of \(I(X)\).

Definition 2.2

Let \(A, B \in W(X)\), \(\alpha \in [0, 1]\) and \(CP(X)\) be the set of all nonempty compact subsets of \(X\). Then

\[p_\alpha(A, B) = \inf_{x \in A_{\alpha}, y \in B_{\alpha}} d(x, y)\]

and \(D_\alpha(A, B) = H(A_{\alpha}, B_{\alpha})\), where \(H\) is the Hausdorff metric between two sets in the collection \(CP(X)\). We define the following functions

\[p(A, B) = \sup_{\alpha} p_\alpha(A, B), \delta(A, B) = \sup_{\alpha} \delta_\alpha(A, B)\text{ and }D(A, B) = \sup_{\alpha} D_\alpha(A, B)\]

It is noted that \(p_\alpha\) is nondecreasing function of \(\alpha\).

Definition 2.3

Let \(A, B \in W(X)\). Then \(A\) is said to be more accurate than \(B\) (or \(B\) includes \(A\)), denoted by \(A \subseteq B\), iff \(A(x) \leq B(x)\) for each \(x \in X\). The relation \(\subseteq\) induces a partial order on \(W(X)\).

Definition 2.4

Let \(X\) be an arbitrary set and \(Y\) be a metric linear space. \(F\) is said to be a fuzzy mapping iff \(F\) is a mapping from the set \(X\) into \(W(Y)\), i.e., \(F(x) \in W(Y)\) for each \(x \in X\). The following proposition is used in the sequel.
Proposition 2.1

([4]). If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.

Following Beg and Ahmed [10], let $(X, d)$ be a metric space. We consider a subcollection of $I(X)$ denoted by $W(X)$. Each fuzzy set $A \in W(X)$, its $\alpha$-level set is a nonempty compact subset (non-fuzzy) of $X$ for each $\alpha \in (0, 1]$. It is obvious that each element $A \in W(X)$ leads to $A \in W(X)$ but the converse is not true.

The authors [10] introduced the improvements of the lemmas in Heilpern [3] as follows.

Lemma 2.1
If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_* (x_0, B) \leq D_0 (A, B)$ for each $B \in W^*(X)$.

Lemma 2.2
$p_* (x, A) \leq d(x, y) + p_* (y, A)$ for all $x, y \in X$ and $A \in W^*(X)$.

Lemma 2.3
Let $x \in X, A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_* (x, A) = 0$ for each $\alpha \in (0, 1]$.

Lemma 2.4
Let $(X, d)$ be a complete metric space, $F: X \rightarrow W^*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

Remark 2.1

It is clear that Lemma 2.4 is a generalization of corresponding lemma in Arora and Sharma [11] and Proposition 3.2 in Lee and Cho [7].

Let $\Psi$ be the family of real lower semi-continuous functions $F: [0, \infty]^6 \rightarrow R, R := \text{the set of all real numbers}$, satisfying the following conditions:

$(\psi_1)$ $F$ is non-increasing in 3rd, 4th, 5th, 6th coordinate variable;

$(\psi_2)$ there exists $h \in (0, 1)$ such that for every $u, v \geq 0$
with

$(\psi_3)$ $F(u, v, u, v, u + v, 0) \leq 0$ or

$(\psi_3)$ $F(u, v, u, v, 0, u + v) \leq 0$, we have $u \leq h \, v$, and

$(\psi_3)$ $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

III. RESULTS AND DISCUSSION


Theorem 3.1

Let $(X, d)$ be a complete metric space and $T_1, T_2$ be fuzzy mappings from $X$ into $W(X)$. If there is a constant $q$, $0 \leq q < 1$, such that, for each $x, y \in X$,

$D(T_1(x), T_2(y)) \leq q \max \{d(x, y), p(x, T_1(x)), p(y, T_2(y)), p(x, T_2(y)), p(y, T_1(x))\}$,

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Remark 3.1

If there is a constant $q$, $0 \leq q < 1$, such that, for each $x, y \in X$,

$D(T_1(x), T_2(y)) \leq q \max \{d(x, y), p(x, T_1(x)), p(y, T_2(y)), p(x, T_2(y)), p(y, T_1(x))\}$,

then the conclusion of Theorem 3.1 remains valid. This result is considered as a special case of Theorem 3.1.

Beg and Ahmed [10] generalized Theorem 3.1 as follows.

Theorem 3.2

Let $(X, d)$ be a complete metric space and $T_1, T_2$ be fuzzy mappings from $X$ into $W^*(X)$. If there is a $F \in \Psi$ such that, for all $x, y \in X$,

$F(D(T_1(x), T_2(y)), d(x, y), p(x, T_1(x)), p(y, T_2(y)), p(x, T_2(y)), p(y, T_1(x))) \leq 0$,

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

We give another different generalization of Theorem 3.1 with contractive condition (1) as follows.

Theorem 3.3

Let $(X, d)$ be a complete metric space and $T_1, T_2$ be fuzzy mappings from $X$ into $W^*(X)$. Assume that there exist $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ such that, for all $x, y \in X$,

$D^3(T_1(x), T_2(y)) \leq c_1 \max \{d^2(x, y), p^2(x, T_1(x)), p^2(y, T_2(y))\} + c_2 \max \{p(x, T_1(x))p(x, T_2(y))\}$, p(y, T_2(y))$,
\[ T_1(x_0)p(y, T_2(y)) + c_3p(x, T_2(y))p(y, T_1(x_0)). \]

Then there exists \( z \in X \) such that \( \{z\} \subset T(z) \) and \( \{z\} \subset T_2(z) \).

**Proof**

Let \( x_0 \) be an arbitrary point in \( X \). Then by **Lemma 2.4**, there exists an element \( x_1 \in X \) such that \( \{x_1\} \subset T(x_0) \). For \( x_1 \in X \), \((T_2(x_1))\) is nonempty compact subset of \( X \). Since \((T_2(x_0)), (T_2(x_1)) \subset CP(X) \) and \( x_1 \in (T_2(x_1)) \), then **Proposition 2.1** asserts that there exists \( x_2 \in (T_2(x_1)) \) such that \( d(x_1, x_2) \leq D_1(T(x_1), T_2(x_1)) \).

So, we obtain from the inequality \( D(A, B) > D(A, B) \forall x \in [0, 1] \) that

\[ d^2(x_1, x_2) \leq D^2(T(x_0), T_2(x_1)) \leq D^2(T(x_0), T_2(x_1)) \]

\[ \leq c_1\max\{d^2(x_0, x_1), p^2(x_0, T_1(x_0)), p^2(x_1, T_2(x_1))\} + c_2\max\{p(x_0, T_1(x_0))p(x_0, T_2(x_1)), p(x_1, T_1(x_0))p(x_1, T_2(x_1))\} \]

\[ + c_3p(x_0, T_2(x_1))(x_1, T_1(x_0)) + c_4\max\{d^2(x_0, x_1), d^2(x_1, x_2)\} + c_2d(x_0, x_1)[d(x_0, x_1) + d(x_1, x_2)]. \]

If \( d(x_1, x_2) > d(x_0, x_1) \), then we have

\[ d^2(x_1, x_2) \leq (c_1 + 2c_2) d^2(x_1, x_2), \]

which is a contradiction. Thus,

\[ d(x_1, x_2) \leq h^2d(x_0, x_1), \]

where \( h^2 = c_1 + 2c_2 < 1 \). Similarly, one can deduce that \( d(x_2, x_3) \leq h^2d(x_1, x_2) \).

By induction, we have a sequence \( (x_n) \) of points in \( X \) such that, for all \( n \in N \cup \{0\} \),

\[ \{x_{2n-1}\} T_1(x_{2n}), \{x_{2n}\} T_2(x_{2n-1}). \]

It follows by induction that \( d(x_n, x_{n+1}) \leq h^2d(x_n, x_1) \). Since

\[ d(x_n, x_m) \leq h^2d(x_n, x_{n+1}) + h^2d(x_{n+1}, x_{n+2}) + \ldots + h^2d(x_m, x_1), \]

\[ \leq h^2 \frac{h^n}{1-h} d(x_0, x_1), \]

then \( \lim_{n \to \infty} d(x_n, x_0) = 0 \). Therefore, \( (x_n) \) is a Cauchy sequence. Since \( X \) is complete, then there exists \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). Next, we show that \( \{z\} \subset T(z) \), \( i = 1, 2 \). Now, we get from **Lemmas 2.1 and 2.2** that

\[ p_d(z, T(z)) \leq d(z, x_{2n+1}) + p_c(z, x_{2n+1}), T_2(z) \leq d(z, x_{2n+1}) + D_d(T_1(x_{2n}), T_2(z)), \]

for each \( a \in [0, 1] \). Taking supremum on \( a \) in the last inequality, we obtain that

\[ p(z, T(z)) \leq d(z, x_{2n+1}) + D(T_1(x_{2n}), T_2(z)). \]

From the inequality (3), we have that

\[ D_2(T_1(x_{2n}), T_2(z)) \leq c_1\max\{d^2(x_{2n}, z), p^2(x_{2n}, T_1(x_{2n})), p^2(z, T_2(z))\} + c_2\max\{p(x_{2n}, T_1(x_{2n}))(x_{2n}, T_2(z)), p(z, T_1(x_{2n}))(p(z, T_2(z))\}

\[ + c_3p(x_{2n}, T_2(z))(x_{2n}, T_2(z)) \leq c_1\max\{d^2(x_{2n}, z), d^2(x_{2n}, x_{2n+1}), p^2(z, T_2(z))\} + c_4\max\{d(x_{2n}, x_{2n+1})(x_{2n}, T_2(z)), d(x_{2n}, x_{2n+1})(p(z, T_2(z))\} + c_5p(x_{2n}, T_2(z))(x_{2n}, x_{2n+1}). \]

(5)

Letting \( n \to \infty \) in the inequalities (4) and (5), it follows that

\[ p(z, T(z)) \leq c_1p(z, T(z)). \]

Since \( c_1 < 1 \), we see that \( p(z, T(z)) = 0 \). So, we get from **Lemma 2.3** that \( \{z\} \subset T(z) \). Similarly, one can be shown that \( \{z\} \subset T_2(z) \).

**Remark 3.2**

(I) Condition (3) is not deducible from condition (2) since the function \( F \) from \([0, \infty)^6 \) into \([0, \infty) \) defined as

\[ F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 + c_1\max\{t_1, t_2, t_3, t_4, t_5, t_6\} - c_3t_3t_6, \]

for all \( t_1, t_2, t_3, t_4, t_5, t_6 \in [0, \infty) \), where \( c_1, c_2, c_3 \in [0, \infty) \) with \( c_1 + c_2 < 1 \) and \( c_2 + c_3 < 1 \), does not generally satisfy condition (3). Indeed, we have that

\[ F(u, u, 0, u, u) = u^2 - c_1u^2 - c_2u^2, \]

for all \( u > 0 \) and does not imply that \( F(u, u, 0, u, u) > 0 \) for all \( u > 0 \).

It suffices to consider \( c_1 = \frac{3}{4} \), \( c_2 = \frac{1}{9} \), \( c_3 = \frac{1}{2} \) and then \( c_1 + c_2 < 1 \) and \( c_2 + c_3 < 1 \) but \( F(u, u, 0, u, u) < 0 \) for all \( u > 0 \).

Therefore, **Theorems 3.2 and 3.3** are two different generalizations of **Theorem 3.1** with contractive condition (1).

(II) If there exist \( c_1, c_2, c_3 \in [0, \infty) \)

with \( c_1 + 2c_2 < 1 \) and \( c_2 + c_3 < 1 \) such that, for all \( x, y \in X \),

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\[ \delta^2(T_1(x), T_2(y)) \leq c_1 \max \{d^2(x, y), p^2(x, T_1(x)), p^2(y, T_2(y)) \} \]
+ \[c_2 \max \{p(x, T_1(x))p(x, T_2(y)), p(y, T_1(x))p(y, T_2(y)) \} \]
+ \[c_3 p(x, T_2(y))p(y, T_1(x)) \]

then the conclusion of Theorem 3.3 remains valid. This result is considered as a special case of Theorem 3.3 because \[D(F_1(x), F_2(y)) \leq \delta(F_1(x), F_2(y)).\] Moreover, this result generalizes Theorem 3.3 of Park and Jeong [8].

**Theorem 3.4**

Let \((T_n: n \in N \cup \{0\})\) be a sequence of fuzzy mappings from a complete metric space \((X, d)\) into \(W^*(X)\). Assume that there exist \(c_1, c_2, c_3 \in [0, \infty)\) with \(c_1 + 2c_2 < 1\) and \(c_2 + c_3 < 1\) such that, for all \(x, y \in X\),

\[ D^2(T_0(x), T_n(y)) \leq c_1 \max \{d^2(x, y), p^2(x, T_0(x)), p^2(y, T_0(y)) \} \]
+ \[c_2 \max \{p(x, T_0(x))p(x, T_n(y)), p(y, T_0(x))p(y, T_n(y)) \} \]
+ \[c_3 p(x, T_n(y))p(y, T_0(x)) \quad \forall n \in N. \]

Then there exists a common fixed point of the family \((T_n: n \in N \cup \{0\})\).

**Proof**

Putting \(T_1 = T_0\) and \(T_2 = T_n \forall n \in N\) in Theorem 3.3. Then, there exists a common fixed point of the family \((T_n: n \in N \cup \{0\})\).

**IV. REFERENCES**