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Fixed Point Theorems in Fuzzy Metric Spaces

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ABSTRACT

In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).
Keywords : Fixed point, Fuzzy metric spaces, Fuzzy mapping.

I. INTRODUCTION

In 1965, the theory of fuzzy sets was investigated by Zadeh [2]. In 1981, Heilpern [3] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [4]. Therefore, several fixed point theorems for types of fuzzy contractive mappings have appeared (see, for instance [1,5–9]).

In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).

II. METHODS AND MATERIAL

Basic Preliminaries

The definitions and terminologies for further discussions are taken from Heilpern [3]. Let (X, d) be a metric linear space. A **fuzzy set** in X is a function with domain X and values in [0, 1]. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the **grade of membership** of x in A. The collection of all fuzzy sets in X is denoted by I(X).

Let $A \in I(X)$ and $\alpha \in [0, 1]$. The ***α*-level set** of A, denoted by A_α, is defined by

$$A_\alpha=\{x:A(x)\geqslant\alpha\} \text{ if } \alpha\in(0,1],\ A_0=\overline{\{x : A(x) > 0\}},$$

whenever $\overline{B}$ is the closure of set (non-fuzzy) B.

Definition 2.1

A fuzzy set A in X is an **approximate quantity** iff its α-level set is a nonempty compact convex subset (non-fuzzy) of X for each $\alpha \in [0, 1]$ and $sup_{x\in X} A(x) = 1$.
The set of all approximate quantities, denoted by $W(X)$, is a subcollection of I(X).

Definition 2.2

Let $A, B \in W(X)$, $\alpha \in [0, 1]$ and $CP(X)$ be the set of all nonempty compact subsets of X. Then

$$p_\alpha(A, B)= \inf_{x\in A_\alpha, y\in B_\alpha} d(x, y),\ \delta_\alpha(A, B) = \sup_{x\in A_\alpha, y\in B_\alpha} d(x, y)$$

and $D_\alpha(A, B)=H(A_\alpha, B_\alpha)$,
where H is the **Hausdorff metric** between two sets in the collection $CP(X)$. We define the following functions

$$p(A, B)=\sup_\alpha p_\alpha(A, B),\ \delta(A, B)= \sup_\alpha \delta_\alpha(A,B) \text{ and } D(A, B)= \sup_\alpha D_\alpha(A, B).$$

It is noted that p_α is nondecreasing function of α.

Definition 2.3

Let $A, B \in W(X)$. Then A is said to be **more accurate** than B (or B includes A), denoted by $A \subset B$, iff $A(x) \leqslant B(x)$ for each $x \in X$.
The relation $\subset$ induces a partial order on $W(X)$.

Definition 2.4

Let X be an arbitrary set and Y be a metric linear space. F is said to be a **fuzzy mapping** iff F is a mapping from the set X into $W(Y)$, i.e., $F(x) \in W(Y)$ for each $x \in X$.
The following proposition is used in the sequel.

Proposition 2.1

([4]). *If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leqslant H(A, B)$.*

Following Beg and Ahmed [10], let (X, d) be a metric space. We consider a subcollection of I(X) *denoted by $W_*(X)$. Each fuzzy set $A \in W_*(x)$, its α-level set is a nonempty compact subset (non-fuzzy) of X for each $\alpha \in [0, 1]$. It is obvious that each element $A \in W(X)$ leads to $A \in W_*(X)$ but the converse is not true.*

The authors [10] introduced the improvements of the lemmas in Heilpern [3] as follows.

Lemma 2.1

If $\{x_0\} \subset A$ for each $A \in W_*(X)$ and $x_0 \in X$, then $p_\alpha(x_0, B) \leqslant D_\alpha(A, B)$ for each $B \in W_*(X)$.

Lemma 2.2

$p_\alpha(x, A) \leqslant d(x, y) + p_\alpha(y, A)$ for all $x, y \in X$ and $A \in W_*(X)$.

Lemma 2.3

Let $x \in X$, $A \in W_*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.

Lemma 2.4

Let (X, d) be a complete metric space, $F: X \to W_*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

Remark 2.1

It is clear that Lemma 2.4 is a generalization of corresponding lemma in Arora and Sharma [1] and Proposition 3.2 in Lee and Cho [7].

Let Ψ be the family of real lower semi-continuous functions $F: [0, \infty)^6 \to R$, $R :=$ the set of all real numbers, satisfying the following conditions:

(ψ_1) F is non-increasing in 3rd, 4th, 5th, 6th coordinate variable,

(ψ_2) there exists $h \in (0, 1)$ such that for every $u, v \geqslant 0$ with

(ψ_{21}) $F(u, v, v, u, u + v, 0) \leqslant 0$ or

(ψ_{22}) $F(u, v, u, v, 0, u + v) \leqslant 0$, we have $u \leqslant h\, v$, and

(ψ_3) $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

III. RESULTS AND DISCUSSION

In 2000, Arora and Sharma [1] proved the following result.

Theorem 3.1

Let (X, d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W(X)$. If there is a constant q, $0 \leqslant q < 1$, such that, for each $x, y \in X$,

$D(T_1(x), T_2(y)) \leqslant q \max\{d(x, y), p(x,T_1(x)), p(y,T_2(y)), p(x,T_2(y)), p(y,T_1(x))\}$,

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Remark 3.1

If there is a constant q, $0 \leqslant q < 1$, such that, for each $x, y \in X$,

$$D(T_1(x), T_2(y)) \leqslant q \max\{d(x, y), p(x,T_1(x)), p(y,T_2(y))\}, \quad (1)$$

then the conclusion of Theorem 3.1 remains valid. This result is considered as a special case of Theorem 3.1. Beg and Ahmed [10] generalized Theorem 3.1 as follows.

Theorem 3.2

Let (X, d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W_*(X)$. If there is a $F \in \Psi$ such that, for all $x, y \in X$,

$$F(D(T_1(x), T_2(y)), d(x, y), p(x, T_1(x)), p(y, T_2(y)), p(x, T_2(y)), p(y, T_1(x))) \leqslant 0, \quad (2)$$

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

We give another different generalization of Theorem 3.1 with contractive condition (1) as follows.

Theorem 3.3

Let (X, d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W_*(X)$. Assume that there exist c_1, c_2, $c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ such that, for all $x, y \in X$,

$D^2(T_1(x), T_2(y)) \leqslant c_1 \max\{d^2(x, y), p^2(x, T_1(x)), p^2(y, T_2(y))\} + c_2 \max\{p(x, T_1(x))p(x, T_2(y)), p(y,$

$T_1(x))p(y, T_2(y))\} + c_3 p(x, T_2(y))p(y, T_1(x)).$ (3)

Then there exists $z \in X$ *such that* $\{z\} \subset T_1(z)$ *and* $\{z\} \subset T_2(z)$.

Proof

Let x_0 be an arbitrary point in X. Then by Lemma 2.4, there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$. For $x_1 \in X$, $(T_2(x_1))_1$ is nonempty compact subset of X. Since $(T_1(x_0))_1$, $(T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0))_1$, then Proposition 2.1 asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1,x_2) \leqslant D_1(T_1(x_0), T_2(x_1))$. So, we obtain from the inequality $D(A, B) \geqslant D_\alpha(A, B)$ $\forall \alpha \in [0, 1]$ that

$$\begin{aligned} d^2(x_1, x_2) &\leqslant D_1^2(T_1(x_0), T_2(x_1)) \\ &\leqslant D^2(T_1(x_0), T_2(x_1)) \\ &\leqslant c_1 \max\{d^2(x_0, x_1), p^2(x_0,T_1(x_0)), p^2(x_1, T_2(x_1))\} \\ &+c_2 \max\{p(x_0, T_1(x_0))p(x_0, T_2(x_1)), p(x_1, T_1(x_0))p(x_1, T_2(x_1))\} \\ &+c_3 p(x_0, T_2(x_1))p(x_1, T_1(x_0)) \\ &+c_1 \max\{d^2(x_0, x_1), d^2(x_1, x_2)\}+c_2 d(x_0, x_1)[d(x_0, x_1)+d(x_1, x_2)]. \end{aligned}$$

If $d(x_1, x_2) > d(x_0, x_1)$, then we have

$d^2(x_1, x_2) \leq (c_1+2c_2)\, d^2(x_1, x_2)$,

which is a contradiction. Thus,

$d(x_1, x_2) \leq h d(x_0, x_1)$,

where $h=c_1+2c_2<1$. Similarly, one can deduce that

$d(x_2, x_3) \leq h d(x_1, x_2)$.

By induction, we have a sequence (x_n) of points in X such that, for all $n \in N \cup \{0\}$,

$\{x_{2n+1}\} T_1(x_{2n})$, $\{x_{2n+2}\} T_2(x_{2n+1})$.

It follows by induction that $d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$. Since

$$d(x_n, x_m) \leq d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})+\ldots+d(x_{m-1}, x_m) \leq h^n d(x_0, x_1)+h^{n+1} d(x_0, x_1)+\ldots+h^{m-1} d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1),$$

then $\lim_{n, m\to\infty} d(x_n, x_m) = 0$. Therefore, (x_n) is a Cauchy sequence. Since X is complete, then there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset T_i(z)$, $i = 1, 2$. Now, we get from Lemmas 2.1 and 2.2 that

$p_\alpha(z,T_2(z)) \leq d(z, x_{2n+1})+p_\alpha(x_{2n+1}, T_2(z)) \leq d(z, x_{2n+1})+D_\alpha(T_1(x_{2n}), T_2(z))$,

for each $\alpha \in [0, 1]$. Taking supremum on α in the last inequality, we obtain that

$$p(z,T_2(z)) \leq d(z, x_{2n+1})+D(T_1(x_{2n}), T_2(z)). \quad (4)$$

From the inequality (3), we have that

$$\begin{aligned} D_2(T_1(x_{2n}),T_2(z)) &\leq c_1 \max\{d^2(x_{2n}, z), p^2(x_{2n}, T_1(x_{2n})), p^2(z, T_2(z))\} \\ &+c_2 \max\{p(x_{2n}, T_1(x_{2n}))p(x_{2n},T_2(z)), p(z,T_1(x_{2n}))p(z,T_2(z))\} \\ &+c_3 p(x_{2n},T_2(z))p(z,T_1(x_{2n})) \\ &\leq c_1 \max\{d^2(x_{2n}, z), d^2(x_{2n}, x_{2n+1}), p^2(z, T_2(z))\} \\ &+c_2 \max\{d(x_{2n}, x_{2n+1})p(x_{2n}, T_2(z)), d(z,x_{2n+1})p(z,T_2(z))\} \\ &+c_3 p(x_{2n}, T_2(z))d(z,x_{2n+1}). \end{aligned} \quad (5)$$

Letting $n \to \infty$ in the inequalities (4) and (5), it follows that

$p(z,T_2(z)) \leq c_1 p(z,T_2(z))$.

Since $c_1<1$, we see that $p(z, T_2(z)) = 0$. So, we get from Lemma 2.3 that $\{z\} \subset T_2(z)$. Similarly, one can be shown that $\{z\} \subset T_1(z)$.

Remark 3.2

(I) Condition (3) is not deducible from condition (2) since the function F from $[0, \infty)^6$ into $[0, \infty)$ defined as

$$F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^2-c_1 \max t_2^2,t_3^2,t_4^2-c_2 \max\{t_3t_5, t_6t_4\}-c_3t_5t_6,$$

for all $t_1, t_2, t_3, t_4, t_5, t_6 \in [0, \infty)$, where $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$, does not generally satisfy condition (ψ_3). Indeed, we have that

$F(u, u, 0, 0, u, u)=u^2-c_1u^2-c_3u^2$, for all $u > 0$ and does not imply that $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

It suffices to consider $c_1=\frac{3}{4}$, $c_2=\frac{1}{9}$, $c_3=\frac{1}{2}$ and then $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ but $F(u, u, 0, 0, u, u) < 0$ for all $u > 0$. Therefore, Theorems 3.2 and 3.3 are two different generalizations of Theorem 3.1 with contractive condition (1).

(II) If there exist $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ such that, for all $x, y \in X$,

$\delta^2(T_1(x), T_2(y)) \leq c_1\max\{d^2(x, y), p^2(x,T_1(x)), p^2(y,T_2(y))\}$
$+c_2\max\{p(x,T_1(x))p(x,T_2(y)), p(y,T_1(x))p(y,T_2(y))\}$
$+c_3p(x,T_2(y))p(y,T_1(x))$,

then the conclusion of Theorem 3.3 remains valid. This result is considered as a special case of Theorem 3.3 because $D(F_1(x), F_2(y)) \leq \delta(F_1(x), F_2(y))$. Moreover, this result generalizes Theorem 3.3 of Park and Jeong [8].

Theorem 3.4

Let (T_n: n $N \cup \{0\}$) be a sequence of fuzzy mappings from a complete metric space (X, d) into $W^*(X)$. Assume that there exist $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ such that, for all $x, y \in X$,

$D^2(T_0(x),T_n(y)) \leq c_1\max\{d^2(x, y),p^2(x,T_0(x)),p^2(y,T_n(y))\}$
$+c_2\max\{p(x,T_0(x))p(x,T_n(y)),p(y,T_0(x))p(y,T_n(y))\}$
$+c_3p(x,T_n(y))p(y,T_0(x)) \ \forall n\in N$.
Then there exists a common fixed point of the family (T_n: n $N \cup \{0\}$).

Proof

Putting $T_1 = T_0$ and $T_2 = T_n \ \forall n \in N$ in Theorem 3.3. Then, there exists a common fixed point of the family (T_n: $n \in N \cup \{0\}$).

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