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Some Common Fixed Point Results for Noncommuting Mappings in Fuzzy Cone Metric Spaces

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ABSTRACT

The existence of coincidence point and common fixed point for noncommuting mappings satisfying certain contractive condition in fuzzy cone metric space is established and result is justified by a counter example. By using this result, some common fixed point theorems are also established for generalize contractive conditions.

Keywords : Fuzzy real number, Fuzzy cone metric space, weakly compatible mappings, Coincidence point and Common fixed point.

I. INTRODUCTION

The idea of cone metric space and cone normed linear space are recent development in functional analysis. The idea of cone metric space was introduced by Long-Guang and Xian [1]. The definition of cone normed linear space is introduced by Samanta et al. [2] and Eshaghi Gordji et al. [3]. Although the concept of cone normed linear space in [2,3] are almost similar. In earlier papers [4,5], the author introduced the idea of fuzzy cone metric space as well as fuzzy cone normed linear space and studied some basic results.

The study of common fixed points of mappings satisfying certain contractive conditions is now a vigorous research activity. In 1976, Jungck [6], proved a common fixed point theorem for commuting mappings, generalizing the Banach contraction principle. After that, different authors developed more results regarding common fixed point theorem by using different types of contractive conditions for noncommuting mappings in metric spaces (for references please see [7–10]. On the other hand Abbas and Jungck [11] developed common fixed point results for noncommuting mappings in cone metric spaces.

In this paper, the existence of coincidence points and common fixed point for noncommuting mappings satisfying some contractive conditions in fuzzy cone metric spaces are established and the results are justified by an example.

Some Preliminary Results

A fuzzy number is a mapping $x:R\rightarrow[0,1]$ over the set R of all reals.

A fuzzy number x is convex if $x(t)\geqslant\min(x(s), x(r))$ where $s\leqslant t\leqslant r$.

If there exists $t_0\in R$ such that $x(t_0)=1$, then x is called normal. For $0<\alpha\leqslant 1$, α-level set of an upper semi continuous convex normal fuzzy number (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha=-\infty$ and $b_\alpha=+\infty$ are admissible. When $a_\alpha=-\infty$, for instance, then $[a_\alpha,b_\alpha]$ means the interval $(-\infty,b_\alpha]$. Similar is the case when $b_\alpha=+\infty$.

A fuzzy number x is called non-negative if $x(t)=0$, $\forall\, t<0$.

Kaleva (Felbin) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by $E(R(I))$ and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering "$\preceq$" in E is defined by $\eta\preceq\delta$ if and only if $a_\alpha^1 \leqslant a_\alpha^2$ and $b_\alpha^1 \leqslant b_\alpha^2$ for all $\alpha\in(0,1]$ where $[\eta]_\alpha=[a_\alpha^1, b_\alpha^1]$ and $[\delta]_\alpha=[a_\alpha^2, b_\alpha^2]$. The strict inequality in E is defined by $\eta\prec\delta$ if and only if $a_\alpha^1 < a_\alpha^2$ and $b_\alpha^1 < b_\alpha^2$ for each $\alpha\in(0,1]$.

Proposition 1.1

[12] *Let* η, $\delta\in E(R(I))$ and $[\eta]_\alpha=[a_\alpha^1, b_\alpha^1]$, $[\delta]_\alpha=[a_\alpha^2, b_\alpha^2]$, $\alpha\in(0,1]$.*Then*

$[\eta\oplus\delta]_\alpha=[a_\alpha^1+a_\alpha^2,\ b_\alpha^1+b_\alpha^2]$

$[\eta\ominus\delta]\alpha=a_\alpha^1-b_\alpha^2,\ b_\alpha^1-a_\alpha^2]$

$[\eta\odot\delta]\alpha=[a_\alpha^1\ a_\alpha^2,\ b_\alpha^1\ b_\alpha^2]$

Definition 1.1

[13] A sequence $\{\eta_n\}$ in *E* is said to be convergent and converges to η denoted by $\lim_{n\to\infty}\eta_n=\eta$ if $\lim_{n\to\infty} a_\alpha^n=a_\alpha$ and $\lim_{n\to\infty} b_\alpha^n=b_\alpha$ where $[\eta_n]_\alpha=[a_\alpha^n, b_\alpha^n]$ and $[\eta]_\alpha=[a_\alpha, b_\alpha]$ $\forall\ \alpha\in(0,1]$.

Note 1.1 [13] If η, δ∈G(R*(I)) then η⊕δ∈G(R*(I)).

Note 1.2 [13] For any scalar *t*, the fuzzy real number t_η is defined as $t_\eta(s)=0$ if $t = 0$ otherwise $t_\eta(s)=\eta(\frac{s}{t})$.

Definition of fuzzy norm on a linear space as introduced by Felbin is given below:

Definition 1.3

[14] Let *X* be a vector space over *R*.

Let || ||:X→R*(I) and let the mappings

L, U: [0,1]×[0,1]→[0,1] be symmetric, nondecreasing in both arguments and satisfy

L(0, 0)=0 and U(1, 1)=1.

Write $[||x||]_\alpha=\left[\|x\|_\alpha^1,\ \|x\|_\alpha^2\right]$ for x∈X, 0<α⩽1 and suppose for all x∈X, $x\neq\bar{0}$ there exists $\alpha_0\in(0,1]$ independent of *x* such that for all $\alpha\leqslant\alpha_0$,

(A) $\|x\|_\alpha^2<\infty$.

(B) Inf $\|x\|_\alpha^1>0$. The quadruple (X, || ||, L, U) is called a fuzzy normed linear space and || || is a fuzzy norm if

(i) $||x||=\bar{0}$ if and only if $x=\bar{0}$;

(ii) ||rx||=|r|||x||, x∈X, r∈R;

(iii) for all x, y∈X,

(a) whenever $s\leqslant\|x\|_1^1$, $t\leqslant\|y\|_1^1$ and $s+t\leqslant\|x+y\|_1^1$, ||x+y||(s+t)⩾L(||x||(s), ||y||(t)),

(b) whenever $s\geqslant\|x\|_1^1$, $t\geqslant\|y\|_1^1$ and $s+t\geqslant\|x+y\|_1^1$, ||x+y||(s+t)⩽U(||x||(s), ||y||(t))

Remark 1.2

[14] Felbin proved that, If L=⋀(Min) and U=⋁(Max) then the triangle inequality (iii) in Definition 1.3is equivalent to ||x+y||⪯||x||⊕||y||. Further $\|\ \|_\alpha^i$ i=1, 2 are crisp norms on *X* for each α∈(0,1]. In that case we simply denote (X, || ||).

Definition 1.6 [4] Let (E, || ||) be a fuzzy real Banach space (Felbin sense) where || ||:E→R*(I).

Denote the range of || || by E*(I). Thus E*(I)⊂R*(I).

Definition 1.7 [4] A member η∈R*(I) is said to be an interior point if ∃ r>0 such that

$S(\eta, r)=\{\delta\in R^*(I): \eta\ominus\delta\prec\bar{r}\}\subset R^*(I)$.

Set of all interior points of R*(I) is called interior of R*(I).

Definition 1.8 [4] A subset of *F* of E*(I) is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim_{n\to\infty}\eta_n=\eta$ implies η∈F.

Definition 1.9 [4] A subset *P* of E*(I) is called a fuzzy cone if

(i) *P* is fuzzy closed, nonempty and $P\neq\{\bar{0}\}$;

(ii) a, b∈R, a, b⩾0, η, δ∈P⇒aη⊕bδ∈P;

(iii) η∈P and -η∈P⇒ $\eta=\bar{0}$.

Given a fuzzy cone P⊂E*(I), define a partial ordering ⩽ with respect to *P* by η⩽δ iff δ⊖η∈P and η<δ indicates that η⩽δ but η≠δ while η≪δ will stand for δ⊖η∈ Int *P* where Int *P* denotes the interior of *P*.

The fuzzy cone *P* is called normal if there is a number K>0 such that for all η, δ∈E*(I), with $\bar{0}\leqslant\eta\leqslant\delta$ implies η⪯Kδ. The least positive number satisfying above is called the normal constant of *P*.

The fuzzy cone *P* is called regular if every increasing sequence which is bounded from above is convergent. That is if {ηn} is a sequence such that $\eta_1\leqslant\eta_2\leqslant\ldots\ldots\leqslant\eta_n\leqslant\ldots\leqslant\eta$ for some η∈E*(I), then there is δ∈E*(I) such that $\eta_n\to\delta$ as n→∞.

Equivalently, the fuzzy cone *P* is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular fuzzy cone is a normal fuzzy cone.

In the following we always assume that *E* is a fuzzy real Banach (Felbin sense) space, *P* is a fuzzy cone in *E* with Int $P\neq\phi$ and ⩽ is a partial ordering with respect to *P*.

Definition 1.10 [4] Let *X* be a nonempty set. Suppose the mapping d:X×X→E*(I) satisfies

(Fd1) $\bar{0}\leqslant d(x, y)$ ∀x, y∈X and $d(x, y)=\bar{0}$ iff x=y;

(Fd2) d(x, y)=d(y, x) ∀x, y∈X;

(Fd3) d(x, y)⩽d(x, z)⊕d(z, y) ∀x, y, z∈X.

Then d is called a fuzzy cone metric and (X, d) is called a fuzzy cone metric space.

Definition 1.11 [4] Let (X, d) be a fuzzy cone metric space. Let $\{x_n\}$ be a sequence in X and $x\in X$. If for every $c\in E$ with $\bar{0} \ll \|c\|$ there is a positive integer N such that for all $n>N$, $d(x_n, x)\ll\|c\|$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty}x_n=x$.

Lemma 1.2 [4] *Let* (X, d) *be a fuzzy cone metric space and P be a normal fuzzy cone with normal constant K. Let* $\{x_n\}$ *be a sequence in X. If* $\{x_n\}$ *is convergent then its limit is unique.*

Definition 1.12 [4] Let (X, d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If for any $c\in E$ with $\bar{0} \ll \|c\|$, there exists a natural number N such that $\forall$ m, $n>N$, $d(x_n, x_m)\ll\|c\|$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.13 [4] Let (X, d) be a fuzzy cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete fuzzy cone metric space.

Definition 1.14 [11] Let f and g be self mappings defined on a set X. If $w=f(x)=g(x)$ for some $x\in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Proposition 1.2 [11] *Let f and g be weakly compatible self-mappings of a set X. If f and g have a unique point of coincidence* $w=f(x)=g(x)$, *then w is the unique common fixed point of f and g.*

2. Common fixed point theorems:

In this section, some common fixed point results for noncommuting and weakly compatible mappings in fuzzy cone metric spaces are established.

Theorem 2.1. *Let* (X, d) *be a fuzzy cone metric space and P be a fuzzy normal cone with normal constant K. Suppose mappings* $f, g:X\to X$ *satisfy*

$d(fx, fy)\leqslant kd(gx, gy)$ $\forall$ $x, y\in X$ *where* $k\in[0,1)$ *is a constant.*

If the range of g contains the range of f and g(X) *is a complete subspace of X, then f and g have unique point of coincidence in X.*

Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0)=g(x_1)$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n)=g(x_{n+1})$.

Then, $d(gx_{n+1}, gx_n)=d(fx_n, fx_{n-1})\leqslant kd(gx_n, gx_{n-1})\leqslant k^2d(gx_{n-1}, gx_{n-2})\leqslant\ldots.$

i.e. $d(gx_{n+1}, gx_n)\leqslant k^n d(gx_1, gx_0)$.

Then for n>m,

$d(gx_n, gx_m)\leqslant d(gx_n, gx_{n-1})\oplus d(gx_{n-1}, gx_{n-2})\oplus\ldots\ldots\oplus d(gx_{m+1}, gx_m)$ (1)

i.e. $d(gx_n, gx_m)\leqslant(k^{n-1}+k^{n-2}+\ldots\ldots+k^m)d((gx_1, gx_0)$. i.e. $d(gx_n, gx_m)\leqslant \frac{k^m}{1-k} d(gx_1, gx_0)$.

Since P is a normal fuzzy cone with normal constant K we have,

$d(gx_n, gx_m)\preceq \frac{k^m}{1-k} d(gx_1, gx_0)$.

Let $d(gx_n, gx_m)=\|y_{n,m}\|$ and $d(gx_1, gx_0)=\|y_{1,0}\|$ where $y_{n,m}, y_{1,0}\in E$.

Then from above we get,

$\|y_{n,m}\|\preceq \frac{k^m}{1-k}\|y_{1,0}\|$

$\Rightarrow \| y_{n,m} \|_\alpha^1 \leqslant \frac{k^m}{1-k} \|y_{1,0}\|_\alpha^1$ and $\| y_{n,m} \|_\alpha^2 \leqslant \frac{k^m}{1-k} \|y_{1,0}\|_\alpha^2$ $\forall$ $\alpha\in(0,1]\Rightarrow\lim_{m,n\to\infty} \| y_{n,m} \|_\alpha^1 =0$ and $\lim_{m,n\to\infty} \| y_{n,m} \|_\alpha^2 = 0$ $\forall$ $\alpha\in(0,1]\Rightarrow\lim_{m,n\to\infty}\|y_{n,m}\|= \bar{0}$ $\Rightarrow\lim_{m,n\to\infty}d(gx_n, gx_m)= \bar{0}$.

This implies that $\{gx_n\}$ is a Cauchy sequence. Since g(X) is complete, there exists a $q\in g(X)$ such that $gx_n\to q$ as $n\to\infty$. Consequently, we can find $p\in X$ such that $g(p)=q$.

Further, $d(gx_n, fp)=d(fx_{n-1}, fp)\leqslant kd(gx_{n-1}, gp)$.

Again since P is a fuzzy normal cone with normal constant K, we get,

$d(gx_n, fp)\preceq Kkd(gx_{n-1}, gp)$.

Since $d(gx_{n-1}, gp)\to \bar{0}$ as $n\to\infty$, by the same argument as above, it follows that

$d(gx_n, fp)\to \bar{0}$ as $n\to\infty$.

On the other hand, $d(gx_n, gp)\to \bar{0}$ as $n\to\infty$.

Since the limit of a convergent sequence in fuzzy cone metric space is unique, we get $f(p)=g(p)$.

Now we show that, f and g has a unique point of coincidence.

For, assume that there exists another point q in X such that $f(q)=g(q)$.

Now, $d(gq, gp)=d(fq, fp)\leqslant kd(gq, gp)$.

Since P is normal cone with normal constant K we get d(gq, gp)≤Kkd(gq, gp).⇒d(gq, gp)= $\bar{0}$ ⇒gq=gp.

Hence *f* and *g* have a unique point of coincidence. Thus from Proposition 1.2(Abas et al.) it follows that *f* and *g* have a unique common fixed point.

The above theorem is justified by the following example.

Example 2.1

Let (E, || ||′) be a Banach space. Define || ||:E→R*(I) by

$$||x||(t)=\begin{cases} 1 \text{ if } t > \| x \|' \\ 0 \text{ if } t \le \| x \|' \end{cases}$$

Then $[||x||]_\alpha$=[||x||′, ||x||′] ∀ α∈(0,1].

It is easy to verify that,

(i) ||x||= $\bar{0}$ iff x= $\bar{0}$ (ii) ||rx||=|r|||x|| (iii) ||x+y||≤||x||⊕||y||.

Thus (E, || ||) is a fuzzy normed linear space (Felbin sense). Let $\{x_n\}$ be a Cauchy sequence in (E, || ||)

So, $\lim_{m,n\to\infty}||x_n-x_m||=\bar{0}$.⇒$\lim_{m,n\to\infty}||x_n-x_m||'=0$.⇒$\{x_n\}$be a Cauchy sequence in (E, || ||′).

Since (E, || ||′) is complete, ∃ x∈E such that $\lim_{m,n\to\infty}||x_n-x||'=0$.

i.e.$\lim_{n\to\infty}||x_n-x||=\bar{0}$.

Thus (E, || ||) is a real fuzzy Banach space.

Define P={η∈E*(I): η⪰C}.

(i) *P* is fuzzy closed.

For, consider a sequence $\{\delta_n\}$ in *P* such that $\lim_{n\to\infty}\delta_n\to\delta$.

i.e.$\lim_{n\to} \delta^1_{n,\alpha}=\delta^1_\alpha$ and $\lim_{n\to} \delta^2_{n,\alpha}=\delta^2_\alpha$ where $[\delta_n]_\alpha$= [$\delta^1_{n,\alpha}$, $\delta^2_{n,\alpha}$] and $[\delta]_\alpha$=[$\delta^1_{n,\alpha}$, $\delta^2_{n,\alpha}$] ∀ α∈(0,1].

Now δ_n⪰ $\bar{0}$ ∀ n.

So, $\delta^1_{n,\alpha}$ ⩾0 and $\delta^2_{n,\alpha}$ ⩾0 ∀ α∈(0,1].⇒$\lim_{n\to}\delta^1_{n,\alpha}$ ⩾0 and $\lim_{n\to}\delta^2_{n,\alpha}$ ⩾0 ∀ α∈(0,1]⇒ $\delta^1_{n,\alpha}$ ⩾0 and $\delta^2_{n,\alpha}$ ⩾0 ∀ α∈(0,1].⇒δ⪰ $\bar{0}$.

So δ∈P. Hence *P* is fuzzy closed.

(ii) It is obvious that, a, b∈R, a, b⩾0, η,δ∈P⇒aη⊕bδ∈P.

(iii) Let η∈P. If -η∈P, then for all t<0 we have (-η)(t)=η(-t)=η(s)⩾0 for s(=-t)>0.

If η(s)=0 ∀ s>0 then η= $\bar{0}$. Otherwise for some s(=-t)>0, η(s)>0.

i.e. for some t<0, (-η)(t)>0. In that case -η does not belong to *P*.

Hence η∈P and -η∈P implies η= $\bar{0}$. Thus *P* is a fuzzy cone in *E*.

Define || ||:X×X→R*(I) by (X=R)

$$d(x, y)(t)=\begin{cases} \frac{|x-y|}{t} \text{ if } t \ge |x-y| \\ 0 \text{ if } t < |x-y| \end{cases}$$

Then $[d(x, y)]_\alpha$=[|x-y|, $|x-y|_\alpha$] ∀ α∈(0,1].

It can be verified that *d* is a fuzzy cone metric (if we chose the ordering ⩽ as ⪯) and thus (X, d) is a fuzzy cone metric space.

Define two functions f, g:X→X by

$$f(x)=\begin{cases} \frac{1}{1+\beta}x \text{ if } x \ne 0 \\ 0 \text{ if } x = 0 \end{cases} \quad \text{and} \quad g(x)=\begin{cases} x \text{ if } x \ne 0 \\ \gamma \text{ if } x = 0 \end{cases}$$

where β⩾1 and γ≠0.

Now we show that d(fx, fy)≤kd(gx, gy) ∀ x, y∈X where k= $\frac{1}{\beta}$ ∈(0,1].

For, d(fx, fy)= ($\frac{1}{\beta+1}x$, $\frac{1}{\beta}y$).

Then [d(fx, fy)]α=[αβ+1|x-y|,1β+1|x-y|]∀α∈(0,1].

Thus

$$d^1_\alpha(fx,fy)= \frac{\alpha}{\beta+1}|x-y| \text{ and } d^2_\alpha(fx,fy)= \frac{1}{\beta+1}|x-y| \quad \forall\alpha\in(0,1]. \quad (2)$$

Again d(gx,gy)=d(x,y). So,$[d(gx,gy)]_\alpha=[d(x,y)]_\alpha$=[α|x-y|,|x-y|],∀α∈(0,1].

Thus d^1_α (gx,gy)=α|x-y| and d^2_α (gx,gy)=|x-y| ∀α∈(0,1]. (3)

From (2) and (3) ∀α∈(0,1] we have,

$$d^1_\alpha(fx,fy)= \frac{1}{\beta+1} d^1_\alpha(gx,gy) \leqslant \frac{1}{\beta+1} d^1_\alpha(gx,gy)$$

and

$$d^2_\alpha(fx,fy)= \frac{1}{\beta+1} d^2_\alpha(gx,gy) \leqslant \frac{1}{\beta+1} d^2_\alpha(gx,gy) \leqslant \frac{1}{\beta+1} d^2_\alpha(gx,gy).$$

This implies that d(fx,fy)≤kd(gx,gy) ∀x,y∈X where k= $\frac{1}{\beta}$.

Moreover *f* and *g* have a coincidence point (x=0) in *X*.

We also observe that *f* and *g* do not commute at the coincidence point 0.

For, fg(0)=f(γ)= $\frac{\gamma}{\beta+1}$ and gf(0)=g(γ)= γ.

Thus *f* and *g* are not weakly compatible. Also *f* and *g* have no common fixed point.

Theorem 2.2

Let(X, d)*be a fuzzy cone metric space and P be a fuzzy normal cone with normal constant K. Suppose mappings* f, g:X→X *satisfy the contractive condition*

d(fx, fy)⩽k(d(fx, gx)⨁d(fy, gy)) ∀x, y∈X where k∈[0, $\frac{1}{2}$) is a constant. If the range of g contains the range of f and g(X) is a complete subspace of X, then f and g have a unique coincidence point in X.

Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof

Let x_0 be an arbitrary point in *X*. Choose a point x1∈X such that $f(x_0)=g(x_1)$. This can be done since the range of *g* contains the range of *f*. Continuing this process, having chosen x_n in *X*, we obtain x_{n+1} in *X* such that $f(x_n)=g(x_{n+1})$.

Then, $d(gx_{n+1}, gx_n)=d(fx_n, fx_{n-1})\leqslant k(d(fx_n, gx_n)\oplus d(fx_{n-1}, gx_{n-1}))=k(d(gx_{n+1}, gx_n)\oplus d(gx_n, gx_{n-1}))\Rightarrow d(gx_{n+1}, gx_n)\leqslant hd(gx_n, gx_{n-1})$ where $h=\frac{k}{1-k}$.

For n>m, we have,

$d(gx_n, gx_m)\leqslant d(gx_n, gx_{n-1})\oplus d(gx_{n-1}, gx_{n-2})\oplus\ldots\oplus d(gx_{m+1}, gx_m)$ i.e. $d(gx_n, gx_m)\leqslant \frac{h^m}{1-h}d(gx_1, gx_0)$ from (1) i.e. $d(gx_n, gx_m)\preceq K\frac{h^m}{1-h}d(gx_1, gx_0)$ (since P is a fuzzy normal cone)$\Rightarrow \lim_{m,n\to\infty}d(gx_n, gx_m)\to\bar{0}$ (from Theorem2.1).

So $\{gx_n\}$ is a Cauchy sequence.

Since g(X) is a complete subspace of *X*, there exists *q* in g(X) such that $gx_n\to q$ as $n\to\infty$.

Consequently we can find *p* in *X* such that g(p)=q.

Thus $d(gx_n, fp)=d(fx_{n-1}, fp)\leqslant kd(gx_{n-1}, gp)$

$\Rightarrow d(gx_n, fp)\preceq Kkd(gx_{n-1}, gp)$

$\Rightarrow d_\alpha^1(gx_n, fp)\leqslant Kk\,d_\alpha^1(gx_{n-1}, gp)$

and

$d_\alpha^2(gx_n, fp)\leqslant Kk\,d_\alpha^2(gx_{n-1}, gp)$ ∀α∈(0,1].$\Rightarrow d(gx_n, fp)\to\bar{0}$ as $n\to\infty$.

Also $d(gx_n, gp)\to\bar{0}$ as $n\to\infty$.

The uniqueness of a limit in a fuzzy cone metric space implies that f(p)=g(p).

Now we show that *f* and *g* have a unique point of coincidence.

For, assume that there exists another point *q* in *X* such that fq=gq.

Now, d(gq, gp) = d(fq, fp)⩽k(d(fq, gq)⨁d(fp, gp))⇒d(gq, gp)⪯Kk(d(fq, gq)⨁d(fp, gp))⇒d(gq, gp)= $\bar{0}$ (since fq= gq and fp=gp)⇒gq=gp.

Thus *f* and *g* have unique point of coincidence. Hence by Proposition 1.2, it follows that *f* and *g* have unique common fixed point.

Theorem 2.3

Let (X, d) *be a fuzzy cone metric space and P be a fuzzy normal cone with normal constant* K<1. *Suppose mappings* f, g:X→X *satisfy the contractive condition*

d(fx, fy)<k(d(fx, gy)⨁d(fy, gx)) ∀x, y∈X *where* k∈[0, $\frac{1}{2}$) *is a constant. If the range of g contains the range of f and* g(X) *is a complete subspace of X, then f and g have a unique coincidence point in X.*

Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof

Let x_0 be an arbitrary point in *X*. Choose a point x_1 in *X* such that $f(x_0)=g(x_1)$. This can be done, since the range of *g* contains the range of *f*. Continuing this process, having chosen $x_n\in X$, we obtain $x_{n+1}\in X$, such that $f(x_n)=g(x_{n+1})$.

Then, $d(gx_{n+1}, gx_n)=d(fx_n, fx_{n-1})\leqslant k(d(fx_n, gx_{n-1})\oplus d(fx_{n-1}, gx_n))$

i.e. $d(gx_{n+1}, gx_n)\leqslant k(d(gx_{n+1}, gx_n)\oplus d(gx_n, gx_{n-1}))\oplus d(gx_n, gx_n))$

i.e. $d(gx_{n+1}, gx_n)\leqslant hd(gx_n, gx_{n-1})$ where $h=\frac{k}{1-k}$.

Now for n>m we get,

$d(gx_n, gx_m)\leqslant d(gx_n, gx_{n-1})\oplus d(gx_{n-1}, gx_{n-2})\oplus\ldots\oplus d(gx_{m+1}, gx_m)$ i.e. $d(gx_n, gx_m)\leqslant(h^{n-1}+h^{n-2}+\ldots+h^m)\,d(gx_1, gx_0)$ i.e. $d(gx_n, gx_m)\leqslant \frac{h^m}{1-h}d(gx_1, gx_0)$

Now by same argument as Theorem 2.2, we obtain a point of coincidence of *f* and *g*. Now we show that *f* and *g* have a unique point of coincidence. For this, assume that there exists *p* and *q* in *X* such that fp=gp and fq=gq.

Now d(gq, gp)=d(fq, fp)⩽k(d(fq, gp)⨁d(fp, gq))

i.e. d(gq, gp)⩽2kd(gq, gp)⇒d(gq, gp)⪯2Kkd(gq, gp) (since P is normal)⇒ $d_\alpha^1(gq, gp)\leqslant 2Kk\,d_\alpha^1(gq, gp)$ and

d_{α}^{2}(gq, gp)⩽2Kkd_{α}^{2}(gq, gp) ∀α∈(0,1]⇒ d_{α}^{1}(gq, gp)=0 and d_{α}^{2}(gq, gp=0 ∀α∈(0,1] (since K<1)⇒d(gq, gp)= $\bar{0}$ ⇒gq=gp.

So point of coincidence is unique. Again from Proposition 1.2, it follows that *f* and *g* have unique common fixed point if they are compatible.

II. CONCLUSION

In this paper, the existence of coincidence points and common fixed points for noncommuting mappings satisfying some contractive conditions are established in fuzzy cone metric spaces and the results are verified by an example. It is an attempt to develop fixed point theorems for noncommuting mappings using different types of contractive conditions. I think that there is a wide scope of research to develop fixed point results in fuzzy cone metric spaces.

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