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Integral Problems of Trigonometric Functions

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ABSTRACT

The article considers six types of integrals related with the powers of trigonometric functions. We can obtain the infinite series expressions of these integrals by using Taylor series expansions and integration term by term theorem. Moreover, we propose some integrals to do calculation and evaluate some definite integrals practically. On the other hand, Maple is used to calculate the approximations of these definite integrals and their infinite series expressions for verifying our answers.
Keywords: Integrals, Trigonometric Functions, Infinite Series Expressions, Taylor Series Expansions, Integration Term by Term Theorem, Maple

I. INTRODUCTION

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics, there are many methods to solve the indefinite integrals including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This paper considers the following six types of integrals related with the powers of trigonometric functions, which are not easy to obtain their answers using the methods mentioned above.

$$\int \theta \cos\theta \sin^r \theta d\theta, \tag{1}$$

$$\int \theta \sin\theta \cos^r \theta d\theta, \tag{2}$$

$$\int \theta \sec^2 \theta \tan^r \theta d\theta, \tag{3}$$

$$\int \theta \csc^2 \theta \cot^r \theta d\theta, \tag{4}$$

$$\int \theta \tan\theta \sec^{r+1} \theta d\theta, \tag{5}$$

$$\int \theta \cot\theta \csc^{r+1} \theta d\theta, \tag{6}$$

where r, θ are real numbers. The infinite series expressions of these integrals can be obtained mainly using Taylor series expansions and integration term by term theorem; these are the major results of this article (i.e., Theorems 1-3). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Moreover, Yu [4-31], Yu and Chen [32], and Yu and Sheu [33-35] used complex power series method, integration term by term theorem, Parseval's theorem, area mean value theorem, and generalized Cauchy integral formula to evaluate some types of integral problems. In this paper, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

II. PRELIMINARIES AND RESULTS

Formulas and Theorems:

The followings are the Taylor series expansions of six inverse trigonometric functions:

2.1.1 Inverse sine function

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1) n! n!} x^{2n+1} \text{, where}$$

$|x| \le 1$.

2.1.2 Inverse cosine function

$$\cos^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1) n! n!} x^{2n+1} \text{, where}$$

$|x| \le 1$.

2.1.3 Inverse tangent function

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{, where } |x| \le 1.$$

2.1.4 Inverse cotangent function

$$\cot^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{, where } |x| \le 1.$$

2.1.5 Inverse secant function

$$\sec^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1) n! n!} x^{-2n-1},$$

where $|x| \ge 1$.

2.1.6 Inverse cosecant function

$$\csc^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1) n! n!} x^{-2n-1} \text{, where}$$

$|x| \ge 1$.

2.1.7 Integration term by term theorem ([36, p269]): Suppose that $\{g_n\}_{n=0}^{\infty}$ is a sequence of Lebesgue integrable functions defined on I. If $\sum_{n=0}^{\infty} \int_I |g_n|$ is convergent, then $\int_I \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} \int_I g_n$.

In the following, we determine the infinite series expressions of the integrals (1) and (2).

Theorem 1 *Suppose that r, θ are real numbers, and r is not a negative even integer, then*

$$\int \theta \cos\theta \sin^r \theta d\theta$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1)(2n+r+2) n! n!} \sin^{2n+r+2} \theta + C, \tag{7}$$

where $-\pi/2 \le \theta \le \pi/2$ *and* $\sin^r \theta$ *exists.*

$$\int \theta \sin\theta \cos^r \theta d\theta = \frac{-\pi}{2(r+1)} \cos^{r+1} \theta$$

$$+ \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1)(2n+r+2) n! n!} \cos^{2n+r+2} \theta + C, \tag{8}$$

where $r \ne -1$, $0 \le \theta \le \pi$ *and* $\cos^r \theta$ *exists.*

Proof $\int \theta \cos\theta \sin^r \theta d\theta$

$= \int x^r \sin^{-1} x dx$ (where $x = \sin\theta$)

$$= \int \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1) n! n!} x^{2n+r+1} dx$$

(by Formula 2.1.1)

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1)(2n+r+2) n! n!} x^{2n+r+2} + C$$

(by integration term by term theorem)

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1)(2n+r+2) n! n!} \sin^{2n+r+2} \theta + C.$$

On the other hand,

$$\int \theta \sin\theta \cos^r \theta d\theta$$

$= -\int x^r \cos^{-1} x dx$ (where $x = \cos\theta$)

$$= -\int \left(\frac{\pi}{2} x^r - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1) n! n!} x^{2n+r+1} \right) dx$$

(by Formula 2.1.2)

$$= \frac{-\pi}{2(r+1)} x^{r+1} + \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1)(2n+r+2) n! n!} x^{2n+r+2} + C$$

(by integration term by term theorem)

$$= \frac{-\pi}{2(r+1)} \cos^{r+1} \theta + \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1)(2n+r+2) n! n!} \cos^{2n+r+2} \theta + C.$$

q.e.d.

Using the same proof as Theorem 1, we can easily obtain the infinite series expressions of the integrals (3), (4), (5) and (6) respectively.

Theorem 2 *If the assumptions are the same as Theorem 1, then*

$$\int \theta \sec^2\theta \tan^r \theta d\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+r+2)} \tan^{2n+r+2}\theta + C, \tag{9}$$

where $-\pi/4 \le \theta \le \pi/4$ *and* $\tan^r\theta$ *exists.*

$$\int \theta \csc^2\theta \cot^r \theta d\theta = \frac{-\pi}{2(r+1)}\cot^{r+1}\theta + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+r+2)} \cot^{2n+r+2}\theta + C, \tag{10}$$

where $r \ne -1$, $\pi/4 \le \theta \le 3\pi/4$ *and* $\cot^r\theta$ *exists.*

Theorem 3 *If* r, θ *are real numbers, and* r *is not a non-negative even integer, then*

$$\int \theta \tan\theta \sec^{r+1}\theta d\theta = \frac{\pi}{2(r+1)}\sec^{r+1}\theta - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(2n+1)(-2n+r)n!n!}\sec^{-2n+r}\theta + C, \tag{11}$$

where $r \ne -1$, $0 \le \theta \le \pi$, $\theta \ne \pi/2$ *and* $\sec^r\theta$ *exists.*

$$\int \theta \cot\theta \csc^{r+1}\theta d\theta = -\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(2n+1)(-2n+r)n!n!}\csc^{-2n+r}\theta + C, \tag{12}$$

where $-\pi/2 \le \theta \le \pi/2$, $\theta \ne 0$ *and* $\csc^r\theta$ *exists.*

III. EXAMPLES

In the following, for the six types of integrals in this paper, we will propose some examples and use Theorems 1-3 to obtain their infinite series expressions. On the other hand, we use Maple to calculate the approximations of some definite integrals and their solutions for verifying our answers.

Example 1 By Eq. (7), we have

$$\int_{-\pi/3}^{\pi/4} \theta \cos\theta \sin^8 \theta d\theta = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(2n+1)(2n+10)n!n!} \times \left[\left(\sin\frac{\pi}{4}\right)^{2n+10} - \left(\sin\frac{-\pi}{3}\right)^{2n+10}\right]. \tag{13}$$

Next, we use Maple to verify the correctness of Eq. (13).

> evalf(int(theta*cos(theta)*(sin(theta))^8,theta=-Pi/3..Pi/4),18);

−0.0240652927573701435

>evalf(sum((2*n)!/(4^n*(2*n+1)*(2*n+10)*n!*n!)*((sin(Pi/4))^(2*n+10)-(sin(-Pi/3))^(2*n+10)),n=0..infinity),18);

−0.0240652927573701436

On the other hand, using Eq. (8) yields

$$\int_{\pi/6}^{2\pi/3} \theta \sin\theta \cos^{10} \theta d\theta = \frac{-\pi}{22}\left[\left(\cos\frac{2\pi}{3}\right)^{11} - \left(\cos\frac{\pi}{6}\right)^{11}\right] + \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(2n+1)(2n+12)n!n!}\left[\left(\cos\frac{2\pi}{3}\right)^{2n+12} - \left(\cos\frac{\pi}{6}\right)^{2n+12}\right]. \tag{14}$$

We also use Maple to verify the correctness of Eq. (14).

>evalf(int(theta*sin(theta)*(cos(theta))^10,theta=Pi/6..2*Pi/3),18);

0.0121788362110693046

>evalf(-Pi/22*((cos(2*Pi/3))^11-(cos(Pi/6))^11)+sum((2*n)!/(4^n*(2*n+1)*(2*n+12)*n!*n!)*((cos(2*Pi/3))^(2*n+12)-(cos(Pi/6))^(2*n+12)),n=0..infinity),18);

0.0121788362110693045

Example 2 It follows from Eq. (9) that

$$\int_{-\pi/6}^{\pi/8} \theta \sec^2\theta \tan^6 \theta d\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+8)} \times \left[\left(\tan\frac{\pi}{8}\right)^{2n+8} - \left(\tan\frac{-\pi}{6}\right)^{2n+8}\right]. \tag{15}$$

Using Maple to verify the correctness of Eq. (15) as follows:

>evalf(int(theta*(sec(theta))^2*(tan(theta))^6,theta=-Pi/6..Pi/8),22);

$$-0.1321337536017807318229$$

>evalf(sum((-1)^n/((2*n+1)*(2*n+8))*((tan(Pi/8))^(2*n+8)-(tan(-Pi/6))^(2*n+8)),n=0..infinity),22);

$$-0.001321337536017807318214$$

In addition, by Eq. (10) we obtain

$$\int_{\pi/3}^{5\pi/9}\theta\csc^2\theta\cot^4\theta d\theta$$
$$=\frac{-\pi}{10}\left[\left(\cot\frac{5\pi}{9}\right)^5-\left(\cot\frac{\pi}{3}\right)^5\right]$$
$$+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+6)}\left[\left(\cot\frac{5\pi}{9}\right)^{2n+6}-\left(\cot\frac{\pi}{3}\right)^{2n+6}\right]. \quad (16)$$

>evalf(int(theta*(csc(theta))^2*(cot(theta))^4,theta=Pi/3..5*Pi/9),18);

$$0.01448449912686310379294$$

>evalf(-Pi/10*((cot(5*Pi/9))^5-(cot(Pi/3))^5)+sum((-1)^n/((2*n+1)*(2*n+6))*((cot(5*Pi/9))^(2*n+6)-(cot(Pi/3))^(2*n+6)),n=0..infinity),18);

$$0.01448449912686310379290$$

Example 3 Using Eq. (11) yields

$$\int_{\pi/9}^{\pi/4}\theta\tan\theta\sec^6\theta d\theta$$
$$=\frac{\pi}{12}\left[\left(\sec\frac{\pi}{4}\right)^6-\left(\sec\frac{\pi}{9}\right)^6\right]$$
$$-\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(2n+1)(-2n+5)n!n!}\left[\left(\sec\frac{\pi}{4}\right)^{-2n+5}-\left(\sec\frac{\pi}{9}\right)^{-2n+5}\right]. \quad (17)$$

We employ Maple to verify the correctness of Eq. (17).

>evalf(int(theta*(tan(theta))*(sec(theta))^6,theta=Pi/9..Pi/4),22);

$$0.7178214890303578935149$$

>evalf(Pi/12*((sec(Pi/4))^6-(sec(Pi/9))^6)-sum((2*n)!/(4^n*(2*n+1)*(-2*n+5)*n!*n!)*((sec(Pi/4))^(-2*n+5)-(sec(Pi/9))^(-2*n+5)),n=0..infinity),22);

$$0.7178214890303578935174$$

On the other hand, by Eq. (12) we have

$$\int_{\pi/8}^{3\pi/8}\theta\cot\theta\csc^{12}\theta d\theta$$
$$=-\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(2n+1)(-2n+11)n!n!}$$
$$\times\left[\left(\csc\frac{3\pi}{8}\right)^{-2n+11}-\left(\csc\frac{\pi}{8}\right)^{-2n+11}\right]. \quad (18)$$

Using Maple to verify Eq. (18) as follows:

>evalf(int(theta*cot(theta)*(csc(theta))^12,theta=Pi/8..3*Pi/8),18);

$$3641.86251813817729$$

>evalf(-sum((2*n)!/(4^n*(2*n+1)*(-2*n+11)*n!*n!)*((csc(3*Pi/8))^(-2*n+11)-(csc(Pi/8))^(-2*n+11)),n=0..infinity),18);

$$3641.86251813817794$$

IV. CONCLUSION

In this study, we use Taylor series expansions and integration term by term theorem to solve some types of integrals. In fact, the applications of the two methods are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topics to other calculus and engineering mathematics problems and use Maple to verify our answers.

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