© 2018 IJSRST | Volume 4 | Issue 2 | Print ISSN: 2395-6011 | Online ISSN: 2395-602X
Themed Section: Science and Technology

Fixed Point Theorems for (ε,λ)-Uniformly Locally Generalized Contractions

Dr. M. Ramana Reddy
Associate Professor of Mathematics, Sreenidhi Institute of Science and technology, Hyderabad, Telangana , India

ABSTRACT

In this paper we define a class called (ε,λ) -uniformly locally generalized contractions and establish a fixed point theorem for such contractions.

Keywords : Generalized metric or D*-Metric Space, Open Ball, (ε,λ) -Uniformly Locally Generalized Contraction.

I. INTRODUCTION

Let X be any set. A mapping $f: X \to X$ is a called a **self-map** of X. If f is a self-map of X then we denote $f(x)$ by fx, for brevity. If f and g are self maps of X, their **composition** fg is defined by $(fg)x = fgx$ for all $x \in X$. Also for any self map f of X, its **n^th iterate**, denoted by f^n, is defined inductively by $f^1 = f$ and $f^n = f\ f^{n-1}$ for $n \geq 2$.

If there is an element $z \in X$ such that $f(z) = z$, then z is called a **fixed point** of f. A theorem which gives a set of conditions on f and/or on X under which the self map f of X has a fixed point is generally called a fixed point theorem. Certain fixed point theorems were proved for self maps of metrizable topological spaces also since such spaces, for all practical purposes, can be considered as metric spaces. Dhage [1] has initiated a study of general metric spaces called D-metric spaces. Later several researchers have made a significant contribution to the fixed point theorems of D-metric spaces in [2], [3], [4], [5] and [6].

II. Preliminaries

2.1Definition: Let X be a non-empty set. A function $D^*: X^3 \to [0,\infty)$ is said to be a **generalized metric** or **D*-metric** on X, if it satisfies the following conditions:

(i) $D^*(x,y,z) \geq 0$ for all $x,y,z \in X$

(ii) $D^*(x,y,z) = 0$ if and only if $x = y = z$

(iii) $D^*(x,y,z) = D^*(\sigma(x,y,z))$ for all $x,y,z \in X$,

where $\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$ (iv)

$D^*(x,y,z) \leq D^*(x,y,w) + D^*(w,z,z)$ for all $x,y,z,w \in X$.

The pair (X, D*), where D* is a generalized metric on X is called a **D*-metric space** or a **generalized metric space.**

2.2Definition: Let (X, D*) be a D*-metric space. For $x \in X$ and $r > 0$, the set $B_{D*}(x,r) = \{y \in X : D*(x,y,y) < r\}$ is called the **open ball** of radius r about x.

For example, if X = $\mathbb{R}$ and $D*: \mathbb{R}^3 \to [0,\infty)$ is defined by $D*(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x, y \in X$.

Then $B_{D*}(0,1) = \{ y \in \mathbb{R} : D*(0,y,y) < \}$

$= \{ y \in \mathbb{R} : 2|y| < 1 \}$

$B_{D*}(0,1) = \{ y \in \mathbb{R} : -\frac{1}{2} < y < \frac{1}{2} \} = \left(\frac{1}{2}, \frac{1}{2}\right)$

2.3Definition: A selfmap f of a D*-metric space (X, D*) is called a (ε, λ) **-uniformly locally generalized contraction**, if there is a number q with $0 \le q < 1$ and a positive constant ε, such that

$$D*(fx, fy, fz) \le q.D*(x,y,y) + r.D*(x, fx, fx) + s.D*(y, fy, fy)$$
$$t.\{D*(x, fy, fy) + D*(y, fx, fx)\}$$

for all $x, y \in X$ with $D*(x,y,y) < \varepsilon$, where $\sup_{x,y \in X} \{q + r + s + 2t\} = \lambda < 1$

III. Main Result

3.1 Theorem: Suppose f is a (ε, λ)-uniformly locally generalized contraction of a D*-metric space (X, D*) and X is f-orbitally complete. Then for every $x \in X$, either

(3.1.1) $D*\left(f^s x, f^{s+1}x, f^{s+1}x\right) \ge \varepsilon$ for all integers $s \ge 0$

or

(3.1.2)the sequence $\{f^n x\}$ converges to u, which is a fixed point of f. Also there is no other fixed point $v \in X$ with $D*(u,v,v) < \varepsilon$.

Proof: For any $x \in X$, consider $\{D*\left(f^s x, f^{s+1}x, f^{s+1}x\right)\}_{s=0}^{\infty}$. Then we have either each of the term in this sequence is greater than or equal to ε or for some term in it is less than ε.

In the first case, the alternative of (3.1.1) of the hypothesis holds.

Let for some integer $s = s_0$, $D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right) < \varepsilon$. Since f is a (ε, λ)-uniformly locally generalized contraction and $D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right) < \varepsilon$, we get numbers q, r, s, and t (all depending on x and y) such that

$$D*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right) = D*\left(ff^{s_0}x, ff^{s_0+1}x, ff^{s_0+1}x\right)$$
$$\le q.D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right) + r.D*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$
$$+ s.D*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right)$$
$$+ t\left\{D*\left(f^{s_0}x, f^{s_0+2}x, f^{s_0+2}x\right) + D*\left(f^{s_0+1}x, f^{s_0+1}x, f^{s_0+1}x\right)\right\}$$

$$\leq q.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)+r.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$+s.D^*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right)$$

$$+t\left\{D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)+D^*\left(f^{s_0+1}x, f^{s_0+2}x, f^{s_0+2}x\right)\right\}$$

$$\leq (q+r+t).D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$+(s+t).D^*\left(f^{s_0+1}, f^{s_0+2}, f^{s_0+2}\right)$$

Therefore

$$(1-s-t).D^*\left(f^{s_0+1}, f^{s_0+2}, f^{s_0+2}\right)\leq (q+r+t).D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

This implies that

$$D^*\left(f^{s_0+1}, f^{s_0+2}, f^{s_0+2}\right)\leq \frac{(q+r+t)}{(1-s-t)}.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$\leq \lambda.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

Also we get by repeated use of the above inequality that

$$D^*\left(f^{s_0+p}, f^{s_0+p+1}, f^{s_0+p+1}\right)\leq \lambda.D^*\left(f^{s_0+p-1}x, f^{s_0+p}x, f^{s_0+p}x\right)$$

$$\leq \lambda^2.D^*\left(f^{s_0+p-2}x, f^{s_0+p-1}x, f^{s_0+p-1}x\right)$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\leq \lambda^p.D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

That is, $D^*\left(f^{s_0+p}x, f^{s_0+p+1}x, f^{s_0+p+1}x\right)<\varepsilon$ for every integer $p=0, 1, 2, 3, \ldots$ and hence for $n\geq s_0$, we have

$$D^*\left(f^{n}x, f^{n+p}x, f^{n+p}x\right)\leq D^*\left(f^{n}x, f^{n+1}x, f^{n+1}x\right)+D^*\left(f^{n+1}x, f^{n+2}x, f^{n+2}x\right)$$

$$+\ \ .\ .\ .\ +D^*\left(f^{n+p-1}x, f^{n+p}x, f^{n+p}x\right)$$

$$\leq\left(\lambda^{n-s_0}+\lambda^{n-s_0+1}+\ldots+\lambda^{n-s_0+p-1}\right)D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_{0+1}}x\right)$$

$$\leq\left(\lambda^{n-s_0}+\lambda^{n-s_0+1}+\ldots+\lambda^{n-s_0+p-1}+\ldots\right)D^*\left(f^{s_0}x, f^{s_0+1}x, f^{s_0+1}x\right)$$

$$D^*\left(f^n x, f^{n+p} x, f^{n+p} x\right) \le \frac{\lambda^{n-s_0}}{1-\lambda} D^*\left(f^{s_0} x, f^{s_0+1} x, f^{s_0+1} x\right)$$

$$\to 0 \text{ as } n \to \infty$$

Thus the sequence $\{f^n x\}$ is a Cauchy sequence in a f-orbitally complete D*-metric space (X, D*) and hence there exists $u \in X$ such that

$$u = \lim_{n \to \infty} f^n x = \lim_{n \to \infty} f^{s_0+p}$$

Therefore there is an integer $n_0 > s_0$ such that

$$D^*\left(f^n x, u, u\right) < \varepsilon \text{ for all } n \ge n_0$$

Now

$$D^*\left(fu, ff^n x, ff^n x\right) \le qD^*\left(u, f^n x, f^n x\right) + rD^*\left(u, fu, fu\right) + sD^*\left(f^n x, f^{n+1} x, f^{n+1} x\right)$$

$$+ t\left\{D^*\left(u, f^{n+1} x, f^{n+1} x\right) + D^*\left(f^n x, fu, fu\right)\right\}$$

$$D^*\left(fu, f^{n+1} x, f^{n+1} x\right) \le qD^*\left(u, f^n x, f^n x\right) + rD^*\left(u, f^{n+1} x, f^{n+1} x\right)$$

$$+ rD^*\left(f^{n+1} x, fu, fu\right) + sD^*\left(f^n x, f^{n+1} x, f^{n+1} x\right)$$

$$+ tD^*\left(u, f^{n+1} x, f^{n+1} x\right) + tD^*\left(f^n x, f^{n+1} x, f^{n+1} x\right)$$

$$+ tD^*\left(f^{n+1} x, fu, fu\right)$$

$$\le qD^*\left(u, f^n x, f^n x\right) + (r+t)D^*\left(u, f^{n+1} x, f^{n+1} x\right)$$

$$+ (s+t)D^*\left(f^n x, f^{n+1} x, f^{n+1} x\right)$$

$$+ (r+t)D^*\left(fu, f^{n+1} x, f^{n+1} x\right)$$

$$\le \lambda D^*\left(u, f^n x, f^n x\right) + \lambda D^*\left(u, f^{n+1} x, f^{n+1} x\right)$$

$$+ \lambda D^*\left(f^n x, f^{n+1} x, f^{n+1} x\right) + \lambda D^*\left(fu, f^{n+1} x, f^{n+1} x\right)$$

which gives

$$(1-\lambda)D^*\left(fu, f^{n+1} x, f^{n+1} x\right) \le \lambda\{ D^*\left(u, f^n x, f^n x\right) + D^*\left(u, f^{n+1} x, f^{n+1} x\right)$$

$$+ D^*\left(f^n x, f^{n+1} x, f^{n+1} x\right) \}$$

Therefore

$$D^*\left(fu, f^{n+1}x, f^{n+1}x\right) \le \frac{\lambda}{(1-\lambda)}\left\{D^*\left(u, f^n x, f^n x\right) + D^*\left(u, f^{n+1}x, f^{n+1}x\right)\right.$$

$$\left. + D^*\left(f^n x, f^{n+1}x, f^{n+1}x\right)\right\}$$

Now letting $n \to \infty$, it follows that $D^*(fu,u,u) = 0$ which implies that $fu = u$, showing that the sequence $\{f^n x\}$ converges to some point of X.

To prove the uniqueness of fixed point of f, suppose that $fv = v$ for some $v \in X$ and $D^*(u,v,v) < \varepsilon$. Then

$$D^*(u,v,v) = D^*(fu, fv, fv)$$

$$\le qD^*(u,v,v) + rD^*(u, fu, fu) + sD^*(v, fv, fv)$$

$$+ t\{D^*(u, fv, fv) + D^*(v, fu, fu)\}$$

$$= qD^*(u,v,v) + rD^*(u,u,u) + sD^*(v,v,v)$$

$$+ t\{D^*(u,v,v) + D^*(v,u,u)\}$$

$$= (q+2t)D^*(u,v,v) = \lambda . D^*(u,v,v)$$

which implies that $D^*(u,v,v) = 0$, since $\lambda < 1$ and hence $u = v$, proving the second part of (2.4.5).

3.2Corollary: Suppose f is a (ε, λ)-uniformly locally generalized contraction of a D*-metric space (X, D*) and X is f-orbitally complete. If for every $x \in X$, there is an integer $n(x)$ such that

(3.2.1) $D^*\left(f^{n(x)}x, f^{n(x)+1}x, f^{n(x)+1}x\right) < \varepsilon$

Then f has a unique fixed point, provided any two fixed points u, v of f are such that $D^*(u,v,v) < \varepsilon$.

Also the sequence $\{f^n x\}$ for any $x \in X$ converges to the unique fixed point of f.

Proof: Follows from Theorem 3.1.

IV. REFERENCES

[1]. B. C. Dhage, "Generalized metric spaces and mappings with fixed point", Bulletin of Calcutta Math. Soc. Vol-84, No. 4, pp 329-336(1992)

[2]. B. Ahmad, M. Ashraf and B.E. Rhoades, "Fixed point theorems for expansive mappings in D-metric spaces", Indian J. Pure and Applied Mathematics Vol-30, No.10, pp.1513-1518 (2001).

[3]. B. C. Dhage, "Generalized metric spaces and mappings with fixed point", Bulletin of Calcutta Math. Soc. Vol-84, No. 4, pp 329-336(1992)

[4]. B. C. Dhage, "Common fixed point principle in D-metric spaces", Bulletin of Calcutta Math. Soc. Vol-91, No. 6, pp 475-480 (1999)

[5]. B.E. Rhoades, "A fixed point theorem for generalized metric spaces", Int. J. Math. Sci. 1 (2005).

[6]. B. Singh and R.K. Sharma, "Common fixed points via compatible maps in D-metric space", Radovi Matemeticki 11 (2002); pp 145-153.