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Effect of Initial Phase on Short Time Limit of Noisy Quantum Walk

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ABSTRACT

Following Brun's formalism for decoherent Discrete Time Quantum Walk (DTQW), a fully analytical formulation of first moment of DTQW has been presented when the quantum walker is subjected to a bit-flip noise and the initial state of the walker is considered to be delocalized up to three lattice sites in 1D. It is shown that when the initial state of the walker is also modulated by an initial phase, particular choice of initial phase angle can nullify the effect of the nonlocality in first moment in short time limit.

Keywords : Quantum Information, Decoherence, Discrete Quantum Walk

I. INTRODUCTION

In the past few decades, quantum version of Classical Random Walk (RW), also known as Quantum Walk (QW)[1–8] has gained popularity due to its numerous applications in multidisciplinary subjects like quantum algorithm, quantum biology etc. It is a striking feature of QW that due to effect of quantum interference, at least in 1D, QW spreads exponentially faster than RW [9].Technically this signifies that in QW, the measure of spread rate i.e. variance is found out to be $\sigma^2 \sim t^2$ as compared to RW where $\sigma^2 \sim t$, t is the number of steps taken by the walker . This exponential speed up is one of the reasons why QW on a line (1D) has been studied extensively [10–13]. Past works on QWs have largely discussed issues about long time dynamics of quantum walker under discrete or continuous time framework. Unlike its continuous time counterpart [CTQW][14], a discrete time quantum walk [DTQW][15] is said to have a Position Hilbert Space and an auxiliary Hilbert space, namely Coin Hilbert Space. A 'quantum coin' is thrown in the Coin Hilbert Space and the result of the 'throw' of the coin decides the direction of the movement of quantum walker in the Position Hilbert Space. In 1D, the direction of movement of the walker is either 'left' or 'right' depending on the result of the 'coin throw'. Then the dynamics of the QW is governed by the joint operation of a shift operator which acts on the Position Hilbert Space and a coin operator which 'flips' the coin state after each time step. The effects of decoherence on DTQW is also extensively studied [16–24] under application of special noise operations with probability p per unit time. In most of the previous papers on DTQW, authors have chosen a walker with initial state localized at origin $|0>$. In this work we have considered DTQW under a bit flip noise type model of decoherence and the walker is said to begin with an initial state delocalized over three lattice sites in 1D. The choice of initial states with nonlocality is important because some of the important attributes of QW dynamics like spread or decay rate depends on how the walker starts its walk. We have also considered that the nonlocal initial state is specially modulated by a phase factor. We have analytically calculated the first moment of the walker starting with this three-site delocalized phase modulated initial state. The structure of our paper in the subsequent sections is as follows. In section II, we have briefly reviewed all the prerequisites required for our work concerning DTQW with initial state localized at origin. In section III, following Brun's formalism [25,26], analytical calculation for the first moment has been presented and the fate of the first moment has been considered for special initial phase factors and the results are briefly discussed in the last section.

II. METHODS AND MATERIAL

Let us briefly review the basic details of decoherent DTQW required for our work. Let us assume that for a DTQW on a line, the Position Hilbert Space is H_w and the Coined Hilbert Space is H_c. The Position Hilbert Space is spanned by the position eigenstates $|x>$, where $x = 0, \pm1, \pm2, \pm3$ .....etc. which are the lattice sites on 1D and the Coin Hilbert Space is spanned by two orthogonal vectors $|R>$ and $|L>$. Hence the combined Position Coin Hilbert space is

$$H = H_w \otimes H_c$$

Let us also choose $\hat{P}_R$ and $\hat{P}_L$ as orthogonal projectors on the coined Hilbert space which satisfy $\hat{P}_R + \hat{P}_L = \hat{I}$ where $\hat{I}$ is the identity operator. As normally done in literature of QW, we will choose the coin flip operator $\hat{C}$ to be Hadamard operator $\hat{H}c$ which satisfies following conditions.

$$\hat{H}c\,|R> = \frac{1}{\sqrt{2}}(|R> + |L>)$$

$$\hat{H}c\,|L> = \frac{1}{\sqrt{2}}(|R> - |L>)$$

The shift operator $\hat{S}$ takes the quantum walker to a step 'right' if the coin state is in one of the two basis states ($|R>$ or $|L>$) or 'left' if the coin state is in the other basis state ($|L>$ or $|R>$). Then the form of shift operator $\hat{S}$ can be expressed as

$$\hat{S} = |R><R| \otimes \sum_x |x+1><x| + |L><L| \otimes \sum_x |x-1><x|$$

Hence, the joint operation of the shift operator $\hat{S}$ and coin operator $\hat{C}$ on the combined Hilbert space is

$$\hat{U} = \hat{S}(\hat{I} \otimes \hat{C})$$

If this operator $\hat{U}$ acts on initial state of the quantum walker, it can produce all subsequent states of the QW dynamics. If we consider the initial state of the quantum walk is $|\phi> = |x_0> \otimes |\varphi_0>$, where $|x_0>$ is the initial position state and $|\varphi_0>$ is the initial coin state , then after t steps the quantum walker will be in a state

$$|\phi_t> = \hat{U}_t\,|\phi>$$

Without any loss of generality, we will consider the initial state of the coin is localized at $|R>$.

Let us begin with a DTQW on the line where the walker is considered to start with initial state localized at origin, i.e $|0>$.We will work on Fourier space. Hence, the position state $|x>$of the particle can be written as, in Fourier space,

$$|x> = \int \left(\frac{dk}{2\pi}\right) e^{-ikx}\,|k> \quad (1)$$

The initial localized state can be written as

$$|x_0> = \int \left(\frac{dk}{2\pi}\right) e^{-ikx_0}\,|k> \quad (2)$$

The initial density matrix for position coin space is

$$\rho_0 = |x_0><x_0| \otimes |\varphi_0><\varphi_0| \quad (3)$$

If ρ_t is the density matrix for position coin space after t steps, the probability distribution formula for the quantum walker will be

$$P(x,t) = Tr\,(\,|x><x|\rho_t) \quad (4)$$

Where $Tr\,(.)$ is the Trace operation. In order to calculate the moments of the walker, we need to calculate the following expression.

$$<X^m> = \sum_x x^m\,P(x,t) \quad (5)$$

For $m = 1$, the above expression produces first moment or the mean value of the quantum walk which is of course,

$$< X >= \sum_x x \, \mathrm{P}(x, t) \quad (6)$$

It is the expression (6) we will be interested to calculate in the next section. When the particle is at origin, the equation (1) reduces to

$$|0 >= \int \left(\frac{dk}{2\pi}\right) |k > \quad (7)$$

Unless otherwise specified, the range of integration is from -π to +π and this range will be used throughout our work. As we are considering a decoherent quantum walker here, we will choose here a particular decoherence in the form of bit flip matrix and this form will be used throughout our calculations.

We will choose the decoherence produced by the operators

$$\hat{A}_0 = \sqrt{p}\hat{X} \ , \ \ \hat{A}_1 = \sqrt{p}\hat{I}$$

Where $\hat{X}$ is Pauli σ_x matrix, $\hat{I}$ is the identity matrix and the walk has a probability p per step being measured. Here the form of Pauli σ_x matrix is used a bit flip noise operator.

III. RESULTS AND DISCUSSION

Let us choose our model where the initial state of the walker is delocalised over three lattice sites namely$|0 >$, $|+1>$ and $|-1>$. The walker begins at an initial state which is in equal superposition of the $|0>$, $|+1>$and $|-1>$, namely

$$|x_0 > = \frac{1}{\sqrt{3}}(| - 1 > +|0 > +|1 >) \quad (8)$$

Where $\quad |1 > = \int \left(\frac{dk}{2\pi}\right) e^{-ik}|k >$

And $\quad | - 1 > = \int \left(\frac{dk}{2\pi}\right) e^{ik}|k >$

Let us further generalize our initial states by considering the lattice sites $|+1>$ and $|-1>$ are modulated by phase factors $e^{i\theta}$ and $e^{-i\theta}$. Hence, our model initial state is

$$|x_0 > = \frac{1}{\sqrt{3}}(e^{-i\theta}| - 1 > +\big|0 > +e^{i\theta}\big|1 >) \quad (9)$$

We have already assumed the initial coin state to be $|\varphi_0>$ and the initial density matrix of the joint position -coin space is ρ_0. Inserting the expression for nonlocal initial state (9) of our model in relation (3), the density matrix ρ_t , after t steps , can be expressed as

$$\rho_t = (\tfrac{1}{3}) \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \ g(k, k')|k >< k'| \otimes L_{kk'}^t |\varphi_0 >< \varphi_0 > \quad (10)$$

Where, $g(k, k') = [\, e^{-i(k-k')} + e^{-i(2\theta-k-k')} + e^{-i(\theta-k)} + e^{i(2\theta-k-k'}) + e^{i(k-k')} + e^{-i(\theta-k)} + e^{-i(\theta-k')} + e^{i(\theta-k')} + 1]$

Here, $L_{kk'}^t$ is a super operator. A superoperator is a special operator which acts on one or more operators. The expression for the super operator here is given by

$$L_{kk'}^t = \sum_{n_1 n_2 n_3 \ldots\ldots n_t} U_k \ A_{n_t} \ldots \ldots . U_k A_{n_1} A_{n_1}^\dagger \ U_{k'}^\dagger \ldots . \ldots A_{n_t}^\dagger U_{k'}^\dagger$$

Where U_k in the expression of superoperator is given by

$$U_k = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & -e^{ik} \end{pmatrix}$$

It is well known that this super operator is trace preserving. It is also important here to note that a trace preserving quantum operation can be described by a set of Kraus operators $A_n, n = 1,2,3, \ldots N$ assumed to satisfy [25]

$$\sum_{n=1}^N A_n^\dagger \, A_n = I$$

Now let us use our expression for density matrix (10) in expression (4) to calculate the probability distribution of the walker.

$$P(x,t) = \frac{1}{3}\int \frac{dk}{2\pi}\int \frac{dk'}{2\pi}\ g(k,k')e^{-ix(k-k')}\ Tr(L^t_{kk'}\chi_0) \quad (11)$$

Where by definition $< x|k >< k'|x > = e^{-ix(k-k')}$ and we have denoted the initial coin density matrix $|\varphi_0 >< \varphi_0|$ by χ_0.

Now we are in a position to calculate the moments of the of the quantum walker.

Inserting the expression (11) in equation (5),

$$< X^m > = \frac{1}{3}\sum_x \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi}\ x^m g(k,k')e^{-ix(k-k')}\ Tr(L^t_{kk'}\chi_0) \quad (12)$$

At this stage let us use following property of derivative of Dirac delta function $\delta(x)$. Namely,

$$(2\pi)(-i)^m\delta^m(k-k') = \textstyle\sum_x x^m\ e^{-x(k-k')} \quad (13)$$

Inserting $m = 1$ in equations (12) and (13), we get the expression for first moment from equation (13) which is as follows,

$$< X >= -\frac{i}{3}\sum_x \int \frac{dk}{2\pi}\int dk'\ \ g(k,k')\delta(k - k')\ Tr(L^t_{kk'}\chi_0) \quad (14)$$

We will not use this particular form of the first moment here. In order to get a more usable form of the first moment, let us integrate the above relation(14).To deduce a more usable form of first moment, let us consider a function $f(k,k')$ chosen from above expression (14)

$$f(k,k') = \ g(k,k')\ Tr(L^t_{kk'}\chi_0)$$

If we now integrate the function on the right of dk' in the equation (14) using function $f(k,k')$ by integration by parts method we will require following mathematical relations.

Namely,

(1)Super operator L is Trace preserving, i.e

$$Tr(L^t_{kk'}\chi_0) = \ Tr(\chi_0)$$

(2)Trace of density matrix (operator) is 1

Using above two relations in the integration by parts, our expression for the first moment for the walker with nonlocal initial state (9) and initial phase θ can be simplified as ,

$$< X > =< X_0 > \ + \frac{2}{3}\int \frac{dk}{2\pi}[Cos\ 2(\theta - k) + 2Cos(\theta - k)]\sum_{j=1}^{t} Tr(ZL^j_{kk}\chi_0) \quad (15)$$

Where Z is defined in [25].

This new analytical expression is our main result. In this expression, $< X_0 >$ is the expression for first moment of the walker for localized initial state $|0 >$ [25],

$$< X_0 > = \ \int \frac{dk}{2\pi}\textstyle\sum_{j=1}^{t} Tr(ZL^j_{kk}\chi_0) (16)$$

It is important to observe in our expression (15) that non local contributions for the first moment arise due to inclusion of |1> and |-1> sites in our initial state and the effect of this non locality is encapsulated in the trigonometric factors arising in addition to the local contribution $< X_0 >$. These trigonometric functions are the reminders of $Cos^2\theta$ type pattern of waveform arising due to interference in classical physics. Next, we proceed further to calculate our expression (15) for special values of initial phase factors. We have used MATHEMATICA software for short time limit t = 20.

The super operator L and the submatrix for the super operator for the chosen bit flip noise operator are used in[27].We have mainly followed Brun's formalism [25] to calculate the Trace part of (15) and (16). The superoperator L_{kk} is linear and we can represent it as a matrix acting on space of two-by two operators. A general two-by-two matrix (or operator) can be written as

$$\hat{O} = \ r_0\hat{I} + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3$$

where $\sigma_{1,2,3}$ are Pauli matrices . We then represent $\hat{O}$ by a column vector with components r_0, r_1, r_2 and r_3. To calculate trace part of first moment, in the spirit of equation II.39 of the article [25], the $r_3$component of our result is found. Then r_3 component is multiplied by cosine factors and then integrations are carried out in equation (15).

We have chosen three special values for initial phase θ to calculate (15). Let us consider three special cases of initial states which give us interesting consequences.

Case I: $\theta = (2n + 1)\pi/4$

We have observed that for $\theta = (2n + 1)\pi/4$, where n = 0,1,2,3...the nonlocal contribution to the first moment in (15) becomes zero. Hence for $\theta = (2n + 1)\pi/4$,the first moment for nonlocal initial state (9) becomes equal to first moment for the local initial state (7) and we conclude that at this particular choice of phase angles, phase in the initial state can nullify the effect of nonlocality in first moment and equation (15) reduces to equation (16).

Case II: $\theta = 0$

In this case, we have observed that for $\theta = 0$ the non local first moment (15) is given by $< X >=< X >_0 + F(p)$ where $F(p)$ is some function of the noise level p and the value of $F(p)$
dependent on the time steps.

Case III: $\theta = \pi$

In this case, we have observed that for $\theta = \pi$ the non local first moment (15) is given by $< X >=< X >_0 - F(p)$ where $F(p)$ is again some function of the noise level p and like Case II, the value of $F(p)$ is again dependent on time steps.

IV. CONCLUSION

We have observed that for each time step and for a particular value of noise level p, the value of $F(p)$ in Case II and Case III are exactly same. Here, we would like to comment that the exact form of this function is not important. The important fact here is to observe that for $\theta = 0$ and $\theta = \pi$ phase angles, both the lattice sites |+1> and |−1> in our initial state (9) appear with '+' and '-' signs respectively. This choice, for symmetry reason, produces '+' and '-' signs in case II and case III. This also proves that choice of initial phase doesn't only nullify the effect of nonlocality in the first moment of the quantum walker, it can also decrease or increase the value of it with respect to local initial state. It will be interesting to observe the fate of the probability distributions under these particular initial phase values. A similar interesting change in survival probability of a DTQW was observed with nonlocal initial state and with change in relative phase of the lattice sites [28] although no noise was considered in that particular work. We believe that our result will remain independent under the choice of form of noise operators. Although our result is fairly general, but one can still generalize it by including more lattice sites by hand. And also, following similar mechanism one can go further to calculate second moment and finally the variance of the walker to gain additional insights of the short and long time dynamics and the spread rate.

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