© 2016 IJSRST | Volume 2 | Issue 3 | Print ISSN: 2395-6011 | Online ISSN: 2395-602X
Themed Section: Science and Technology

The Paradoxical Case of Force-Acceleration Transformation in Relativity

A. Sfarti
387 Soda Hall, UC Berkeley

ABSTRACT

During one of my recent classes, an interesting question, never heard before, was posed by one of the students: "How come that the relativistic acceleration transformation transforms zero acceleration into zero acceleration but transforms zero force into non-zero force?" In the current note I will explain this apparent paradox. The proof is not trivial and, to my best knowledge, cannot be found in the literature. The note is intended for undergraduate students and for instructors who teach special relativity, especially the dynamics chapters. PACS: 03.30.+p
Keywords: Lorentz Transformation, Relativistic Force, Acceleration

I. INTRODUCTION

1. The Strange Case of Misleading Newtonian Intuition

Let S and S' be two frames in inertial motion with respect to each other with the velocity V aligned with the x axis, A particle of rest mass m moves with arbitrary velocity $\mathbf{u}=(u_x,u_y,u_z)$ and arbitrary acceleration $\mathbf{a}=(a_x,a_y,a_z)$, as measured in frame S. The force applied on the particle in frame S is $\mathbf{F}=(F_x,F_y,F_z)$. In frame S', the acceleration experienced by the particle is [1-3]:

$$a_x^{'}=\frac{a_x}{\gamma^3(V)(1+\frac{u_xV}{c^2})^3} \qquad (1.1)$$

$$\gamma(V)=\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

Throughout this paper will use the notation established by Tolman [1].

So, for $a_x=0$ it follows that $a_x^{'}=0$. On the other hand, the force experienced by the particle in frame S' is [1-3]:

$$F_x^{'}=F_x+\frac{u_yV}{c^2+u_xV}F_y+\frac{u_zV}{c^2+u_xV}F_z \qquad (1.2)$$

So, for $F_x=0$, $F_x^{'}\neq 0$. This seems very puzzling, since our intuition would expect that null acceleration in one frame would result into null force in **that** frame but this is not the case for frame S'. In other words, our (Newtonian) intuition tells us that $a_x^{'}=0 \Rightarrow F_x^{'}=0$ but this is **not** the case. In order to understand what is really going on we need to remember that in relativity $\mathbf{F}\neq m\mathbf{a}$ but rather $\mathbf{F}=\frac{d\mathbf{p}}{dt}$ where

$\mathbf{p}=\gamma(u)m\mathbf{u}$. Therefore:

$$p_x=\gamma(u)mu_x \qquad (1.3)$$

Implies:

$$F_x=\frac{d}{dt}\gamma(u)mu_x=m\gamma^3(u)[(1-\frac{u_y^2+u_z^2}{c^2})a_x+\frac{u_xu_y}{c^2}a_y+\frac{u_xu_z}{c^2}a_z] \qquad (1.4)$$

There are two possibilities:

a) $u_x=0$ (so the mass is stationary in S at all times)

This means $a_x = 0$ and $p_x = 0$. $a_x = 0$ implies $a_x^{'} = 0$ and $p_x = 0$ implies $F_x = 0$. According to (1.2) $F_x^{'} \neq 0$. This is explained by the fact that:

$$F_x^{'} = \frac{d}{dt'}\gamma(u')mu_x^{'} = m\gamma^3(u')[(1-\frac{u_y^{'2}+u_z^{'2}}{c^2})a_x^{'} + \frac{u_x^{'}}{c^2}(u_y^{'}a_y^{'}+u_z^{'}a_z^{'})] \tag{1.5}$$

We know that:

$$\begin{aligned} u_x^{'} &= V \\ u_y^{'} &= u_y / \gamma(V) \\ u_z^{'} &= u_z / \gamma(V) \end{aligned} \tag{1.6}$$

This means $a_x = 0$ and $p_x = 0$. $a_x = 0$ implies $a_x^{'} = 0$ and $p_x = 0$ implies $F_x = 0$. According to (1.2) $F_x^{'} \neq 0$. This is explained by the fact that:

$$F_y u_y = -F_z u_z \tag{1.7}$$

but this a **particular** situation, **not** the general case. For general F_y, F_z:

$$F_x^{'} = \frac{u_y V}{c^2}F_y + \frac{u_z V}{c^2}F_z \tag{1.8}$$

$$F_x^{'} = m\gamma^3(u')\frac{u_x^{'}}{c^2}(u_y^{'}a_y^{'} + u_z^{'}a_z^{'}) \tag{1.9}$$

One could argue again that $F_x^{'} = 0$ if $u_y^{'}a_y^{'} = -u_z^{'}a_z^{'}$ but this is just a **particular** case, not the general one.

b) $u_x \neq 0$

In this case both $F_x \neq 0$ (by virtue of $p_x \neq 0$) and $F_x^{'} \neq 0$ (by virtue of either (1.2) or (1.5)). Expressions (1.4) and (1.5) demonstrate that the dependency of the force component aligned with one axis (x, in our example) on the accelerations aligned with the transverse axes (y and z, in our example) is an intrinsic effect, not an artifact of the coordinate transformation, as expression (1.2) would lead us to believe.

2. What About the Transverse Forces?

Given the symmetry of the problem it is sufficient to study only the case of the acceleration and force in one direction, for example the y-axis. The transformation formulas are [1,2]:

$$a_y^{'} = \frac{a_y}{\gamma^2(V)(1+\frac{u_x V}{c^2})^2} - \frac{a_x}{\gamma^2(V)(1+\frac{u_x V}{c^2})^3}\frac{u_y V}{c^2} \tag{2.1}$$

$$F_y^{'} = \frac{F_y}{\gamma^2(V)(1+\frac{u_x V}{c^2})} \tag{2.2}$$

In this case, we observe a similar disproof of our Newtonian intuition, $F_y = 0 \Rightarrow F_y^{'} = 0$ does not imply that $a_y = 0 \Rightarrow a_y^{'} = 0$. The reason is similar, $a_y^{'}$ is not only a function of a_y but also a function of a_x

II. CONCLUSION

Starting from an apparent paradox that illustrates a discrepancy between the transformation of force and acceleration in special relativity, we have explained the fact that there is no paradox whatsoever. . In order to understand what is really going on we need to remember that in relativity $\mathbf{F} \neq m\mathbf{a}$ but rather $\mathbf{F} = \frac{d\mathbf{p}}{dt}$

III. REFERENCES

[1] R. Tolman, “Relativity, Thermodynamics and Cosmology”, Dover Publications Inc., (1987) pp. 27, 46

[2] O. Moller, “The Theory of Relativity”, Oxford at the Clarendon Press, 1962

[3] W.G.V. Rosser, “An Introduction to the Theory of Relativity”,(Butterworth), 1967