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Proof of the Conjencture 5.2 Taken From the Research Paper "Research Problems in Number Theory"

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ABSTRACT

The conjecture is taken from the research paper "Research problems in Number Theory" published by Nguyen Cong Hao, Imre K´atai and Bui Minh Phong in the year 2014, that elaborated the concept of additive function, multiplicative function. The conjecture that is proved herein, is deduced for k = 1 from the Wirsing's Theorem that says any real valued multiplicative function/modulus < 1 has a mean value or its limit exists. In this paper, we proved the conjecture 5.2 from the same paper in a simplified way using some basic concepts of arithmetic.
Keywords : Additive Function, Logarithmic Series, Multiplicative Group.

I. INTRODUCTION

Additive functions (mod 1)

Let $T = \mathrm{P}/Z$. We say that $F \in A_T$ (= set of additive functions mapping into *T*) is of finite support if $F\left(p^{\alpha}\right) = 0$ holds for every large prime *p*.

Let $F_0, F_1, \ldots, F_{k-1} \in A_T$, and

$$L_n(F_0, \ldots, F_{k-1}) := F_0(n) + F_1(n+1) + \ldots + F_{k-1}(n+k-1)$$

Conjecture:

If $F_v \in A_T \quad (v = 0, \ldots, k-1)$,

$$L_n(F_0, \ldots, F_{k-1}) \to 0 \qquad (n \to \infty),$$

then there exist suitable real numbers $\tau_0, \ldots, \tau_{k-1}$ *such that* $\tau_0 + \ldots + \tau_{k-1} = 0$,

and if $H_j(n) := F_j(n) - \tau_j \log n$, then

$$L_n(H_0, \ldots, H_{k-1}) = 0 \qquad (n = 1, 2, \ldots).$$

Proof of the conjecture

$$L_n(F_0, \ldots, F_{k-1}) := F_0(n) + F_1(n+1) + \ldots + F_{k-1}(n+k-1)$$

$$F_v \in A_T \quad (v = 0, \ldots, k-1),$$

$$L_n(F_0, \ldots, F_{k-1}) = 0 \qquad (n \to \infty),$$

i.e. $L_n(F_0, \ldots, F_{k-1}) = F_0(n) + F_1(n+1) + \ldots + F_{k-1}(n+k-1) = 0$ (1)

Define that,

$F_0(n) = \tau \log n$ (mod 1), which is restricted to $F_0(n) = \tau \log n$ for a continuous homomorphism from R_x (=multiplicative group of positive real numbers) to N..

Then (1) implies,

$$L_n(F_0,\dots,F_{k-1}) = F_0(n) + F_1(n+1) + \dots + F_{k-1}(n+k-1)$$

$$= \tau_0 \log(n) + \tau_1 \log(n+1) + \dots + \tau_{k-1} \log(n+k-1)$$

$$= \log(n)^{\tau 0} + \log(n+1)^{\tau 1} + \dots + \log(n+k-1)^{\tau k-1}$$

by Logarithm Laws,

$$= \log\left[n^{\tau 0}(n+1)^{\tau 1}\dots(n+k-1)^{\tau k-1}\right]$$

$$= \log\left[n^{\tau 0} n^{\tau 1}\left(1+\frac{1}{n}\right)^{\tau 1}\dots n^{\tau k-1}\left(1+\frac{k-1}{n}\right)^{\tau k-1}\right]$$

$$= \log\left[n^{(\tau 0+\dots+\tau k-1)}\left(1+\frac{1}{n}\right)^{\tau 1}\dots\left(1+\frac{k-1}{n}\right)^{\tau k-1}\right]$$

$$= \log n^{(\tau 0+\dots+\tau k-1)} + \log\left[\left(1+\frac{1}{n}\right)^{\tau 1}\dots\left(1+\frac{k-1}{n}\right)^{\tau k-1}\right]$$

$$= (\tau_0 + \dots + \tau_{k-1})\log n + \log\left[\left(1+\frac{1}{n}\right)^{\tau 1}\dots\left(1+\frac{k-1}{n}\right)^{\tau k-1}\right] \qquad (2)$$

Since, function is restricted R_x to N , i.e. for $n = 1,2,,3,\dots,\infty.$

$$\log n = 2\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{n-1}{n+1}\right)^{2n-1}$$

$$\therefore \log n = 2\left[\frac{1}{1} + \frac{1}{3}\left(\frac{n-1}{n+1}\right)^3 + \frac{1}{5}\left(\frac{n-1}{n+1}\right)^5 + \dots + \frac{1}{2(k-1)-1}\left(\frac{n-1}{n+1}\right)^{2(k-1)-1} + \dots + \infty\right]$$

$$\therefore \log n = 2\left[1 + \frac{1}{3}\left(\frac{1-\frac{1}{n}}{1+\frac{1}{n}}\right)^3 + \dots + \frac{1}{2(k-1)-1}\left(\frac{1-\frac{1}{n}}{1+\frac{1}{n}}\right)^{2(k-1)-1} + \dots + \infty\right] \qquad (3)$$

substituting equation (3) in equation (2), we have

$$L_n(F_0,\dots,F_{k-1}) = F_0(n) + F_1(n+1) + \dots + F_{k-1}(n+k-1) =$$

$$(\tau_0 + \dots + \tau_{k-1})\, 2\left[1 + \frac{1}{3}\left(\frac{1-\frac{1}{n}}{1+\frac{1}{n}}\right)^3 + \dots + \frac{1}{2(k-1)-1}\left(\frac{1-\frac{1}{n}}{1+\frac{1}{n}}\right)^{2(k-1)-1} + \dots + \infty\right]$$

$$+ \log\left[\left(1+\frac{1}{n}\right)^{\tau 1}\dots\left(1+\frac{k-1}{n}\right)^{\tau k-1}\right]$$

Applying $n \to \infty,$ we get

$$(\tau_0 + \dots + \tau_{k-1})2\left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2(k-1)-1} + \dots + \infty\right] + \log(1) = 0$$

$$\because L_n(F_0,\dots,F_{k-1}) \to 0 \qquad (n \to \infty),$$

$$\therefore (\tau_0 + \dots + \tau_{k-1})2\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{2(k-1)-1}+\dots+\infty\right]+0=0$$

$$\Rightarrow (\tau_0 + \dots + \tau_{k-1})2\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{2(k-1)-1}+\dots+\infty\right]=0$$

Since,

$$2\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{2(k-1)-1}+\dots+\infty\right]\neq 0$$

$$\therefore (\tau_0 + \dots + \tau_{k-1}) = 0 \tag{4}$$

Also,

$$L_n(H_0,\dots,H_{k-1}) = H_0 + H_1 + \dots + H_{k-1} \qquad (n = 1,2,\dots).$$

Defined as,

$$H_j(n) := F_j(n) - \tau_j \log n, \qquad \text{(by statement)}$$

$$L_n(H_0,\dots,H_{k-1}) = H_0 + H_1 + \dots + H_{k-1} \qquad (n = 1,2,\dots).$$

$$L_n(H_0,\dots,H_{k-1}) = [F_0(n) - \tau_0 \log(n)] + \dots + [F_{k-1}(n+k-1) - \tau_{k-1}\log(n+k-1)]$$

Logarithmic terms of above equation leads to the following form, from the above explanation,

$$\tau_0 \log(n) + \dots + \tau_{k-1}\log(n+k-1) = (\tau_0 + \dots + \tau_{k-1})2\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{2(k-1)-1}+\dots+\infty\right]$$

$$L_n(H_0,\dots,H_{k-1}) = [F_0(n) + \dots + F_{k-1}(n+k-1)]$$
$$- (\tau_0 + \dots + \tau_{k-1})2\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{2(k-1)-1}+\dots+\infty\right]$$

From (1) and (4),

$$L_n(H_0,\dots,H_{k-1}) = [0] - (0)$$

$$L_n(H_0,\dots,H_{k-1}) = 0$$

Thus, the conjecture is proved.

II. REFERENCES

[1]. Nguyen CH, Imre K and Bui Minh P. (2014). Research Problems in Number Theory. Annales Univ. Sci. Budapest., Sect. Comp. 43: 267-277.