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An Application of the Generalized Bernoulli Equation Method to the Fractional Nonlinear Kawahara Equation

OJO M.O. [1], IJILA P.O. [1], OMOLIKI A.J. [2], KOLEBAJE O.T. [1]*

[1]Department of Physics, Adeyemi College of Education, Ondo, Nigeria
[2]Department of Physics and Solar, Bowen University, Iwo, Nigeria

ABSTRACT

An application of the generalized Bernoulli equation method to the fractional nonlinear fifth order Kawahara equation is presented in this paper. We applied this method to solve the fractional nonlinear Kawahara equation by using the generalized Bernoulli equation which has 13 different known solutions as the auxiliary equation. This method is a simple, reliable and powerful tool for solving the fifth order nonlinear Kawahara equation as it produces an interesting range of solutions.

Keywords: generalized Bernoulli equation method, nonlinear differential equations, travelling wave solutions, Kawahara equation.

I. INTRODUCTION

In Soliton theory, the study of exact solutions to these nonlinear equations plays a very germane role, as they provide much information about the physical models they describe. In recent times, it has been found that many physical, chemical and biological processes are governed by nonlinear partial differential equations of non-integer or fractional order [1-4].

Computer algebraic systems such as Mathematica, Matlab, Maple, and Scilab have made research into methods of obtaining exact solution to NLPDEs an interesting area of research in the last decade. These methods include the Hirota bilinear method,[5] the inverse scattering transform,[6] the Backlund transform,[7-8] the Darboux transform,[9] the sine-cosine method,[10] the tanh-function method,[11-12] the exp-function method,[13] the Homogenous balance method,[14] (G'/G) expansion method, [15] etc.

In this paper, we apply the Generalized Bernoulli equation method for finding exact solution to the fractional nonlinear fifth order Kawahara equation defined in the sense of Jumarie's modified Riemann-Liouville derivatives. The method utilizes the Generalized Bernoulli equation with 13 different solutions as an auxiliary equation.

II. METHODS AND MATERIAL

Description of the Generalized Bernoulli Equation Method

The proposed method can be described as follows: consider a nonlinear PDE with the independent variables x, y, z and t of the form

$$F\left(u, D_t^{\delta}u,\ D_x^{\eta}u,\ D_y^{\sigma}u,\ D_z^{\gamma}u, \dots\right) = 0 \qquad 0 < \alpha, \eta, \sigma, \gamma \leq 1 \qquad (2)$$

F is a polynomial in u and its partial derivatives (integer and fractional) with respect to x, y, z and t, in which the highest order derivatives and the nonlinear terms are also involved.

Step 1: We begin by transforming Eq. (2) into a nonlinear ordinary differential equation (ODE) of integer order by applying a fractional complex transformation proposed by Li and He: [16]

$$u(x, y, z, t) = U(\xi)$$

$$\xi = \frac{x^{\eta}}{\Gamma(1+\eta)} + \frac{y^{\sigma}}{\Gamma(1+\sigma)} + \frac{z^{\gamma}}{\Gamma(1+\gamma)} - \frac{ct^{\delta}}{\Gamma(1+\delta)} \qquad (3)$$

Where c is an arbitrary constant and Eq. (2) reduces to a nonlinear integer order ODE of the form

$$G(u, u', u'', u''', \dots) = 0 \qquad (4)$$

If possible, we integrate term by term one or more times with the integration constants set to zero for simplicity.

Step 2: We assume that the solution to Eq. (2) can be expressed as a polynomial in V

$$U(\xi) = \sum_{i=0}^{m} \alpha_i V^i \qquad \alpha_m \neq 0 \qquad (5)$$

Where $\alpha_0, \alpha_1, \dots, \alpha_m$ are constants to be determined and V satisfies the generalized Bernoulli equation of the form

$$V' = rV + sV^2 \qquad s \neq 0 \qquad (6)$$

Where r and s are arbitrary constants. To determine m, we consider the homogenous balance between the highest order derivative and the highest order nonlinear term(s).

Step 3: Substitute Eq. (5) with the determined value of m into Eq. (4) using Eq. (6), and collect all terms with the same order of V^i for $i = 0, 1, 2, 3, \dots$ together. If the coefficients of V^i vanish separately, we have a set of algebraic equations in $\alpha_0, \alpha_1, \dots, \alpha_m, c, r$ and s that is solved with the aid of Mathematica.

Step 4: Finally, substituting $\alpha_0, \alpha_1, \dots, \alpha_m$, c and the general solutions to Eq. (6) into Eq. (5) yield the exact travelling wave solutions of Eq. (2). The 13 different solutions of the generalized Bernoulli equation under four different classes are presented below with the solutions depending on the nature of r :

For a real and non-zero r, i.e. $r^2 > 0$ and $s \neq 0$, the solutions to Eq. (6) are:

$$V_1 = -\frac{r}{2s}\left[1 + \tanh\left(\frac{r}{2}\xi\right)\right]$$

$$V_2 = -\frac{r}{2s}\left[1 + \coth\left(\frac{r}{2}\xi\right)\right]$$

$$V_3 = -\frac{r}{2s}[1 + \tanh(r\xi) \pm i\,\mathrm{sech}(r\xi)]$$

$$V_4 = -\frac{r}{2s}[1 + \coth(r\xi) \pm \mathrm{csch}(r\xi)]$$

$$V_5 = -\frac{r}{4s}\left[2 + \tanh\left(\frac{r}{4}\xi\right) + \coth\left(\frac{r}{4}\xi\right)\right]$$

$$V_6 = \frac{r}{2s}\left[\frac{\sqrt{J^2 + K^2} - J\cosh(r\xi)}{J\sinh(r\xi) + K} - 1\right]$$

$$V_7 = -\frac{r}{2s}\left[\frac{\sqrt{K^2 - J^2} + J\sinh(r\xi)}{J\cosh(r\xi) + K} + 1\right]$$

J and K are two non-zero real constants that satisfy $K^2 - J^2 > 0$.

$$V_8 = \frac{\pm re^{r\xi}}{s(1 \mp e^{r\xi})}$$

$$V_9 = \frac{\pm re^{r\xi}}{s(i \mp e^{r\xi})}$$

$$V_{10} = \frac{r\left[\sqrt{J^2 - K^2} \pm \left(Je^{r\xi} - iK\right)\right]}{\mp s[J(e^{r\xi} - e^{-r\xi}) - 2iK]}$$

J and K are two non-zero real constants that satisfy $J^2 - K^2 > 0$.

$$V_{11} = \frac{-r\varphi e^{r\xi}}{s(1 + \varphi e^{r\xi})}$$

$$V_{12} = \frac{-re^{r\xi}}{s(\varphi + e^{r\xi})}$$

where φ is an arbitrary constant.

For $r = 0$, $s \neq 0$ and an arbitrary constant μ, the solution to Eq. (5) is:

$$V_{13} = \frac{-1}{s\xi + \mu}$$

III. RESULTS AND DISCUSSION

Application

Consider the space and time fractional fifth order Kawahara equation given by

$$\frac{\partial^{\gamma} u(x,t)}{\partial t} + u(x,t)\frac{\partial^{\sigma} u(x,t)}{\partial x} + \frac{\partial^{3\sigma} u(x,t)}{\partial x^3} - \frac{\partial^{5\sigma} u(x,t)}{\partial x^5} = 0 \qquad (7)$$

We apply the fractional complex transformation given by

$$u(x,t) = U(\xi) \qquad \xi = \frac{x^{\eta}}{\Gamma(1+\eta)} - \frac{ct^{\delta}}{\Gamma(1+\delta)} \qquad (8)$$

where c is an arbitrary constant. By using Eq. (8), Eq. (7) can be reduced into the following integer order ODE:

$$-cU' + UU' + U''' - U''''' = 0 \qquad (9)$$

Integrating Eq. (10) once with respect to ξ and setting the integration constant to zero yields

$$-cU + {U^2}/{2} + U'' - U'''' = 0 \qquad (10)$$

By considering the homogenous balance between the highest order derivative U'''' and the highest order nonlinear term U^2, we deduce that $m = 4$.

Then the solution to Eq. (10) can be written as

$$U(\xi) = \alpha_4 V^4 + \alpha_3 V^3 + \alpha_2 V^2 + \alpha_1 V + \alpha_0 \quad \text{where } \alpha_4 \neq 0 \qquad (11)$$

where $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3$ and α_4 are arbitrary constants to be determined by algebraic calculations. Substituting Eq. (11) and its derivatives into Eq. (10) and collecting all terms with the same power of V together yields a system of algebraic equation presented below

V^0: $\alpha_0{}^2 - 2c\alpha_0 = 0$

V^1: $-c\alpha_1 + r^2\alpha_1 - r^4\alpha_1 + \alpha_0\alpha_1 = 0$

V^2: $6rs\alpha_1 - 30r^3s\alpha_1 + \alpha_1{}^2 - 2c\alpha_2 + 8r^2\alpha_2 - 32r^4\alpha_2 + 2\alpha_0\alpha_2 = 0$

V^3: $2s^2\alpha_1 - 50r^2s^2\alpha_1 + 10rs\alpha_2 - 130r^3s\alpha_2 + \alpha_1\alpha_2 - c\alpha_3 + 9r^2\alpha_3 - 81r^4\alpha_3 + \alpha_0\alpha_3 = 0$

V^4: $-120rs^3\alpha_1 + 12s^2\alpha_2 - 660r^2s^2\alpha_2 + \alpha_2{}^2 + 42rs\alpha_3 - 1050r^3s\alpha_3 + 2\alpha_1\alpha_3 - 2c\alpha_4 + 32r^2\alpha_4 - 512r^4\alpha_4 + 2\alpha_0\alpha_4 = 0$

V^5: $-24s^4\alpha_1 - 336rs^3\alpha_2 + 12s^2\alpha_3 - 1164r^2s^2\alpha_3 + \alpha_2\alpha_3 + 36rs\alpha_4 - 1476r^3s\alpha_4 + \alpha_1\alpha_4 = 0$

V^6: $-240s^4\alpha_2 - 2160rs^3\alpha_3 + \alpha_3{}^2 + 40s^2\alpha_4 - 6040r^2s^2\alpha_4 + \alpha_2\alpha_4 = 0$

V^7: $-360s^4\alpha_3 - 2640rs^3\alpha_4 + \alpha_3\alpha_4 = 0$

V^8: $-1680s^4\alpha_4 + \alpha_4{}^2 = 0$

Case 1 $\alpha_4 = 1680s^4 \quad \alpha_3 = 3360rs^3 \quad \alpha_2 = 1680s^2/13 \quad \alpha_1 = 0 \quad \alpha_0 = -72/169 \quad c = -36/169 \quad r = \pm 1/\sqrt{13}$

Case 2 $\alpha_4 = 1680s^4 \quad \alpha_3 = 3360rs^3 \quad \alpha_2 = 1680s^2/13 \quad \alpha_1 = 0 \quad \alpha_0 = 0 \quad c = 36/169 \quad r = \pm 1/\sqrt{13}$

Case 3

$\alpha_4 = 1680s^4 \quad \alpha_3 = 3360rs^3 \quad \alpha_2 = 280/13\,(-s^2 + 91r^2s^2)$

$$\alpha_1 = 280/13\,(-rs + 13r^3s) \quad \alpha_0 = 0 \quad c = \frac{(31 + 2093r^2)}{1690} \quad r = \pm\sqrt{\frac{-31 \pm 3i\sqrt{31}}{260}}$$

Case 4 $\alpha_4 = 1680s^4 \quad \alpha_3 = 3360rs^3 \quad \alpha_2 = 280/13\,(-s^2 + 91r^2s^2)$

$$\alpha_1 = \frac{280(-rs + 13r^3s)}{13} \quad \alpha_0 = \frac{(-31 - 2093r^2)}{845} \quad c = -\frac{(31 + 2093r^2)}{1690} \quad r = \pm\sqrt{\frac{-31 \pm 3i\sqrt{31}}{260}}$$

Substituting the different sets of solution to the algebraic equation and the general solutions to Eq. (6) into Eq. (11), we obtain 13 different travelling wave solutions of the space and time fractional nonlinear Kawahara equation for each of the sets of solutions. From case 1, we obtain the following exact solutions:

$$U(\xi) = \alpha_4 V^4 + \alpha_3 V^3 + \alpha_2 V^2 + \alpha_1 V + \alpha_0$$

For a real r, the solutions to Eq. (7) are

$$U_1 = \frac{3}{169}\left[-24 + 35\,\mathrm{sech}^4\left(\frac{\xi}{2\sqrt{13}}\right)\right]$$

$$U_2 = \frac{3}{169}\left[-24 + 35\,\mathrm{csch}^4\left(\frac{\xi}{2\sqrt{13}}\right)\right]$$

$$U_3 = \frac{24}{169}\left[-3 - \frac{70e^{\frac{2\xi}{\sqrt{13}}}}{\left(\mp i + e^{\frac{\xi}{\sqrt{13}}}\right)^4}\right]$$

$$U_4 = -24 + 35\left(\frac{e^{\frac{\xi}{\sqrt{13}}}J + K - \sqrt{J^2 + K^2}}{K + J\sinh\left(\frac{\xi}{\sqrt{13}}\right)}\right)^4 + 140\left(\frac{e^{\frac{\xi}{\sqrt{13}}}J + K - \sqrt{J^2 + K^2}}{K + J\sinh\left(\frac{\xi}{\sqrt{13}}\right)}\right)^2 + 140\left(-1 + \frac{\sqrt{J^2 + K^2} - J\cosh\left(\frac{\xi}{\sqrt{13}}\right)}{K + J\sinh\left(\frac{\xi}{\sqrt{13}}\right)}\right)^3$$

$$U_5 = \frac{3}{169}\left[-24 + 140\left(-1 + \frac{\sqrt{J^2+K^2} - J\cosh\left(\frac{\xi}{\sqrt{13}}\right)}{K - J\sinh\left(\frac{\xi}{\sqrt{13}}\right)}\right)^2 + 140\left(-1 + \frac{\sqrt{J^2+K^2} - J\cosh\left(\frac{\xi}{\sqrt{13}}\right)}{K - J\sinh\left(\frac{\xi}{\sqrt{13}}\right)}\right)^3 + 35\left(-1 + \frac{\sqrt{J^2+K^2} - J\cosh\left(\frac{\xi}{\sqrt{13}}\right)}{K - J\sinh\left(\frac{\xi}{\sqrt{13}}\right)}\right)^4\right]$$

$$U_6 = \frac{3}{169}\left[-24 - \frac{70J^2}{\left(K + J\cosh\left(\frac{\xi}{\sqrt{13}}\right)\right)^4}\left(-3J^2 - 4JK\cosh\left(\frac{\xi}{\sqrt{13}}\right) + (J^2 - 2K^2)\cosh\left(\frac{2\xi}{\sqrt{13}}\right) + 4\sqrt{K^2 - J^2}\sinh\left(\frac{\xi}{\sqrt{13}}\right)\left(J + K\cosh\left(\frac{\xi}{\sqrt{13}}\right)\right)\right)\right]$$

$$U_7 = \frac{3}{169}\left[-24 + \frac{70J^2}{\left(K + J\cosh\left(\frac{\xi}{\sqrt{13}}\right)\right)^4}\left(3J^2 - 4JK\cosh\left(\frac{\xi}{\sqrt{13}}\right) + (J^2 - 2K^2)\cosh\left(\frac{2\xi}{\sqrt{13}}\right) + 4\sqrt{K^2 - J^2}\sinh\left(\frac{\xi}{\sqrt{13}}\right)\left(J + K\cosh\left(\frac{\xi}{\sqrt{13}}\right)\right)\right)\right]$$

J and K are two non-zero real constants that satisfy $K^2 - J^2 > 0$.

$$U_8 = -\frac{48\, e^{\frac{2\xi}{\sqrt{13}}}\left(26 + 3\cosh\left(\frac{2\xi}{\sqrt{13}}\right) - 12i\sinh\left(\frac{\xi}{\sqrt{13}}\right)\right)}{169\left(-i + e^{\frac{\xi}{\sqrt{13}}}\right)^4}$$

$$U_9 = -\frac{48\, e^{\frac{2\xi}{\sqrt{13}}}\left(26 + 3\cosh\left(\frac{2\xi}{\sqrt{13}}\right) + 12i\sinh\left(\frac{\xi}{\sqrt{13}}\right)\right)}{169\left(i + e^{\frac{\xi}{\sqrt{13}}}\right)^4}$$

$$U_{10} = \frac{1}{169\left(K + i\,J\sinh\left(\frac{\xi}{\sqrt{13}}\right)\right)^4}\left[840J^2\sqrt{J^2 - K^2}\cosh\left(\frac{\xi}{\sqrt{13}}\right) + 6J^2(41J^2 - 34K^2)\cosh\left(\frac{2\xi}{\sqrt{13}}\right) + 48iJK\,(13J^2 - 6K^2)\sinh\left(\frac{\xi}{\sqrt{13}}\right) + 420iJ^2K\sqrt{J^2 - K^2}\sinh\left(\frac{2\xi}{\sqrt{13}}\right) - 9\left(-67J^4 + 24J^2K^2 + 8K^4 + J^4\cosh\left(\frac{4\xi}{\sqrt{13}}\right) - 8iJ^3K\sinh\left(\frac{3\xi}{\sqrt{13}}\right)\right)\right]$$

$$U_{11}=-\frac{48e^{\frac{4\xi}{\sqrt{13}}}}{169\left(J-e^{\frac{2\xi}{\sqrt{13}}}J+2i\,e^{\frac{\xi}{\sqrt{13}}}K\right)^4}\left[3\left(-67J^4+24J^2K^2+8K^4+J^4\cosh\left(\frac{4\xi}{\sqrt{13}}\right)\right)-8iJK\sinh\left(\frac{\xi}{\sqrt{13}}\right)(29J^2-12K^2)+280J^2\sqrt{J^2-K^2}\cosh\left(\frac{\xi}{\sqrt{13}}\right)\left(J+i\,K\sinh\left(\frac{\xi}{\sqrt{13}}\right)\right)+2J^2\cosh\left(\frac{2\xi}{\sqrt{13}}\right)\left(-41J^2+34K^2-24iJK\sinh\left(\frac{\xi}{\sqrt{13}}\right)\right)\right]$$

$$U_{12}=-\frac{48e^{\frac{4\xi}{\sqrt{13}}}}{169\left(-J+e^{\frac{2\xi}{\sqrt{13}}}J+2i\,e^{\frac{\xi}{\sqrt{13}}}K\right)^4}\left[3\left(-67J^4+24J^2K^2+8K^4+J^4\cosh\left(\frac{4\xi}{\sqrt{13}}\right)\right)+8iJK\sinh\left(\frac{\xi}{\sqrt{13}}\right)(29J^2-12K^2)-280J^2\sqrt{J^2-K^2}\cosh\left(\frac{\xi}{\sqrt{13}}\right)\left(J-i\,K\sinh\left(\frac{\xi}{\sqrt{13}}\right)\right)+2J^2\cosh\left(\frac{2\xi}{\sqrt{13}}\right)\left(-41J^2+34K^2+24iJK\sinh\left(\frac{\xi}{\sqrt{13}}\right)\right)\right]$$

J and K are two non-zero real constants that satisfy $J^2-K^2>0$.

$$U_{13}=\frac{3}{169\left(1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^4}\left[35s^4\left(-1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^4\pm 140s^3\left(-1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^3\left(1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)-24\left(1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^4+140s^2\left(-1+e^{\frac{2\xi}{\sqrt{13}}}\varphi\right)^2\right]$$

$$U_{14}=\frac{3}{169\left(e^{\frac{\xi}{\sqrt{13}}}+\varphi\right)^4}\left[e^{\frac{4\xi}{\sqrt{13}}}(-24+35s^2(2+s)^2)-4e^{\frac{3\xi}{\sqrt{13}}}(24+35s^3(2+s))\varphi+2e^{\frac{2\xi}{\sqrt{13}}}(-72+35s^2(-4+3s^2))\varphi^2-4e^{\frac{\xi}{\sqrt{13}}}(24+35s^3(-2+s))\varphi^3+(-24+35s^2(-2+s)^2)\varphi^4\right]$$

$$U_{15}=\frac{3}{169\left(1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^4}\left[35s^4\left(-1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^4-140s^3\left(-1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^3\left(1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)-24\left(1+e^{\frac{\xi}{\sqrt{13}}}\varphi\right)^4+140s^2\left(-1+e^{\frac{2\xi}{\sqrt{13}}}\varphi^2\right)^2\right]$$

where $\xi=x^{\eta}/\Gamma(1+\eta)+36t^{\delta}/169\,\Gamma(1+\delta)$

Remark 1 The exact travelling wave solutions of the fractional nonlinear Kawahara equation obtained using the generalized Bernoulli equation method for the first set of solution (Case 1) are presented in U_1-U_{15}. The corresponding exact solutions to the fractional nonlinear Kawahara equation for Cases 2-4 can be constructed in a similar way as Case 1. The travelling wave solutions to the fractional nonlinear Kawahara equation obtained were checked by putting them back into Eq. (7) with the aid of Mathematica.

IV. CONCLUSION

With the aid of fractional complex transformation which converts nonlinear fractional partial differential equations into ordinary differential equations of integer order, the fractional nonlinear Kawahara equation was converted into an integer order equivalent. The generalized Bernoulli equation method was then applied to solve the ensuing Kawahara equation. The generalized Bernoulli method is a reliable and efficient method for solving the nonlinear fractional Kawahara equation as it has the ability of producing about 60 different solutions.

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