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On New Difference Sequence Space and Their Generalised Almost Statistical Convergence

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ABSTRACT

The object of the present paper is to introduce the notion of generalised almost statistical (GAS) convergence of bounded real sequences, which generalises the notion of almost convergence as well as statistical convergence of bounded real sequences. We also introduce the concept of Banach statistical limit functional and the notion of GAS convergence mainly depends on the existence of Banach statistical limit functional. We prove the existence of Banach statistical limit functional. Also, the existence GAS convergent sequence, which is neither statistical convergent nor almost convergent. Lastly, some topological properties of the space of all GAS convergent sequences are investigated.

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I. INTRODUCTION & PRELIMINARIES

A sequence (ξ_n) of real number is said to be convergent to a real number l if for $\epsilon > 0 \; \exists \; n_0 \in \mathbb{N}$ such that $|\xi_k - l| < \epsilon \; \forall \; k > n_0$. There are several generalizations of usual convergence, viz. almost convergence (see [13], [4]), statistical convergence (see [5], [7]) etc. Nevertheless, it is always better to have larger set of convergent sequence in more generalized sense under investigation.

The existence of Banach limit functionals was proven by Banach (see [13]) in 1932. Using Banach limits, in 1948, Lorentz (see [4]) introduced the notion of almost convergence, which is a generalization of usual convergence of real sequences. Again in 1951 Fast (see [5]) and Steinhaus (see [7]) introduced independently the notion of statistical convergence by rigorous use of natural density of subsets of $\mathbb{N}$, which is another generalization of usual convergence. Salat (see [15]), Fridy (see [1], [9]), Miller (see [6]) and many others (see [2], [8], [10]) studied the convergence of statistical convergence.

Mursaleen (see [12]) introduced the idea of λ-statistical convergence in 2000. If the sequence λ is chosen, particularly, by $\lambda = (1,2,3,4,\dots)$, then λ-statistical convergence coincides with the statistical convergence. In 2001 Kostyrko et al. (see [11]) introduced the notion of ideal convergence of real sequences. If we consider the ideal of all subsets of $\mathbb{N}$ having natural density zero, then the ideal convergence coincides with statistical convergence. Later Lahiri and Das (see [3]) extended the concept of ideal convergence for nets in topological spaces. Further some generalization of usual convergence were introduced and studied in (see [2], [14], [16]).
In this paper, the extension of a certain type of Banach limit functional is shown which are designated as Banach statistical limit functionals.

With the help of Banach statistical limit functional we have introduced generalized almost statistical (GAS) convergence of bounded real sequences. GAS convergence is a generalization of almost convergence as well as statistical convergence.

At the end, we also investigated some topological properties of the space of all GAS convergent sequences. The scope of this paper is :

Section 2 : Deals with Banach limit functionals, almost convergence and statistical convergence. Section 3 : Deals with the main results of the paper.

II. BANACH LIMIT FUNCTIONALS, ALMOST CONVERGENCE & STATISTICAL CONVERGENCE

The limit functional f on the space c of all convergent real sequences defined by

$f(x) = \lim_{n\to\infty} x_n\,, x \in c$, can be extended to the space l_∞ of all bounded real sequences by Hahn-Banach Extension Theorem, where l_∞ is the normed linear space with sup-norm defined by

$\|x\|_\infty = \sup_{n\in\mathbb{N}}|x_n|\,,\ \forall\, x \in l_\infty$.

Banach (see [13]) showed the existence of certain extensions which are called Banach limits defined as follows :

Definition 2.1

A functional $B : l_\infty \to \mathbb{R}$ is called Banach limit if it satisfies the following :

(i) $\|B\| = 1$,

(ii) $B|_c = f$, where f is the limit functional on c,

(iii) If $x \in l_\infty$ with $x_n \geq 0,\ \forall\, n \in \mathbb{N}$,
then $B(x) \geq 0$,

(iv) If $x \in l_\infty$, then $B(x) = B(Sx)$,
where S is the shift operator defined by
$S((x_n)_{n=1}^{\infty}) = S((x_n)_{n=2}^{\infty})$.

The concept of almost convergence was introduced by Lorentz (see [4]) in 1948 by using Banach limit functionals.

Definition 2.2

For some $x \in l_\infty$, if $B(x)$ is unique (i.e. invariant) for all Banach limit functionals B, then the sequence x is called almost convergent to $B(x)$.

Let $\mathcal{A} \subset l_\infty$ be the set of all almost convergent real sequences. Clearly $c \subsetneq \mathcal{A}$.

Definition 2.3

Let $P \subset \mathbb{N}$.

If the limit

$$\delta(P) = \lim_{n\to\infty} \frac{|P \cap \{1,2,\dots,n\}|}{n}$$

exists, then $\delta(P)$ is called the natural density or asymptotic density of P in $\mathbb{N}$.

Note:

Any subset P of $\mathbb{N}$ may not have natural density as it depends totally on the existence of the limit.

By using the concept of natural density, in 1951, fast (see [5]), Steinhaus (see [7]) introduced independently the notion of statistical convergence, which is another generalisation of usual convergence.

Definition 2.4

A sequence $(p_k)_k$ of real numbers is called convergent statistically to $\ell \in \mathbb{R}$

if for any $\epsilon > 0,\ \delta(\{k \in \mathbb{N} : |p_k - \ell| > \epsilon\}) = 0$.

We use the notation $p_k \xrightarrow{stat} \infty$ or $stat \lim_{n\to\infty} p_n = \ell$.

Example 2.1

The sequence $(\lambda_n)_n$ defined by

$$\lambda_j = \begin{cases} j & \text{if } j = k^2, k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is statistically convergent to 0.

Lemma 2.1

Let $C, D \subset \mathbb{N}$.

If $\delta(C), \delta(D), \delta(C \cup D)$ exists,

Then

$\max\{\delta(C), \delta(D)\} \leq \delta(C \cup D)$
$\leq \min\{\delta(C) + \delta(D), 1\}$

Furthermore, if $\delta(C) = \delta(D) = 0$, then $\delta(C \cup D)$ exists and equals to 0.

Lemma 2.2

Let $(p_n)_n$ be a real sequence.

Then, $p_n \xrightarrow{stat} \ell$ if and only if there exists some

$J \subset \mathbb{N}$ with $J = \{\alpha_1 < \alpha_2 < \cdots < \alpha_n < \ldots\}$
such that $\delta(J) = 1$ and $\lim_{j\to\infty} p_{\alpha j} = \ell$.

This lemma is well known.

Lemma 2.3

Let $(r_n)_n$ be a statistically convergent real sequence and $stat \lim_{n\to\infty} r_n = \ell$.

If $(s_n)_n$ is defined by $s_n = r_n - \ell,\ \forall\, n \in \mathbb{N}$,
then $(s_n)_n$ converges statistically to 0.

Lemma 2.4

Let $(p_n)_n$ be a statistically convergent real sequence.

Then, the telescoping sequence of $(p_n)_n$ is statistically convergent to 0

i.e. $stat \lim_{n\to\infty} (p_n - p_{n+1}) = 0$.

Proof:

Let $stat \lim_{n\to\infty} p_n = l$ and $\epsilon > 0$ be any real number.

Now, for any $n \in \mathbb{N}$,

$$\begin{aligned}|p_{n+1} - p_n| &\le |p_{n+1} - l| + |p_n - l|\\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ &= \epsilon\end{aligned}$$

Using Contrapositive

$$|p_{n+1} - p_n| > \epsilon$$

$$\Rightarrow |p_{n+1} - l| > \tfrac{\epsilon}{2} \text{ or } |p_n - l| > \tfrac{\epsilon}{2}$$

$$\Rightarrow \{n \in \mathbb{N} : |p_{n+1} - p_n| > \epsilon\}$$

$$\subset \left\{n \in \mathbb{N} : |p_{n+1} - l| > \frac{\epsilon}{2}\right\} \cup \left\{n \in \mathbb{N} : |p_n - l| > \frac{\epsilon}{2}\right\}$$

$$\Rightarrow \delta(\{n \in \mathbb{N} : |p_{n+1} - p_n| > \epsilon\}) = 0.$$

$$\Rightarrow stat \lim_{n\to\infty} (p_n - p_{n+1}) = 0.$$

Lemma 2.5

Let $(x_n)_n$ and $(y_n)_n$ be two real sequences such that $x_n \xrightarrow{stat} l$ and $y_k = x_k\ a.a.k.$

Then, $y_n \xrightarrow{stat} l$

Proof:

Let $E = \{k \in \mathbb{N} : y_k \neq x_k\}$ and $\epsilon > 0$ be any real number.

Then $\delta(E) = 0$.

Now, $\{n \in \mathbb{N} : |y_n - \ell| > \epsilon\}$

$$\subset \{n \in \mathbb{N} : |x_n - \ell| > \epsilon\} \cup E$$

$$\Rightarrow \delta(\{n \in \mathbb{N} : |y_n - \ell| > \epsilon\}) = 0.$$

Hence $y_n \xrightarrow{stat} l$.

Definition 2.5

A family I of subsets of $\mathbb{N}$ is said to be an ideal of $\mathbb{N}$ if

(i) $A \cup B \in I$ for each $A, B \in I$

(ii) $A \subset B$ with $B \in I$ implies $A \in I$.

Definition 2.6

A real sequence $(z_k)_k$ is said to be I-convergent to l if for any $\epsilon > 0$, the set $\{k \in \mathbb{N} : |z_k - l| > \epsilon\} \in I$.

Lemma 2.6

Let X be a normed linear space and Y be a subspace of X.

If $\alpha \in X - \bar{Y}$ and

$\mu = d(\alpha, Y) = \inf\{d(\alpha, y) : y \in Y\}$,

then $\exists$ a bounded linear functional $f : X \to \mathbb{R}$ such that $f(\alpha) = 1, f(y) = 0, \forall y \in Y$ with $\|f\| = \mu^{-1}$.

Also, Lorentz (see [4]) characterised the almost convergence given as follows.

Lemma 2.7

Let $x = (x_n)_n \in l_\infty$.

Then $(x_n)_n$ is almost convergent to some ℓ if and only if

$$\lim_{k\to\infty} \frac{x_p + x_{p+1} + \cdots + x_{P+k-1}}{k} = \ell$$

holds for each $p \in \mathbb{N}$.

this lemma is well known.

Example 2.2

The divergent sequence $(1, 0, 1, 0, \ldots) \in l_\infty$ is almost convergent to $\frac{1}{2}$.

But it is not statistically convergent.

Example 2.3

Consider a sequence $x = (x_n)_n$ in $\{0, 1\}$ constructed as follows :

$$x = (\underbrace{0,0,\ldots,0}_{100 \text{ copies}}, \overbrace{1,1,\ldots,1}^{10 \text{ copies}}, \underbrace{\ldots,\ldots,0}_{100^2 \text{ copies}}, \overbrace{1,1,\ldots,1}^{10^2 \text{ copies}},$$
$$\underbrace{0,0,\ldots,0}_{100^3 \text{ copies}}, \overbrace{1,1,\ldots,1}^{10^3 \text{ copies}}, \ldots)$$

Then, $(x_n)_n$ is statistically convergent to 0.

But it is not almost convergent.

We now discuss the main results of the paper.

III. MAIN RESULTS

Salat (see [15]) showed that st is a closed subspace of l_∞.

Since, $\phi_n \xrightarrow{stat} a, \xi_n \xrightarrow{stat} b$

Then we have (i) $\phi_n + \xi_n \xrightarrow{stat} a + b$

(ii) $\lambda\xi_n \xrightarrow{stat} \lambda b,$

for any $\lambda \in \mathbb{R}$ and $\phi, \xi \in st$,

Definition: A function $g : st \to \mathbb{R}$ defined by $g(x) = stat \lim_{n\to\infty} x_n$ is a linear functional on st.

We call this functional g by statistical limit functional on st

Theorem 3.1

The linear functional $g : st \to \mathbb{R}$ defined by $g(x) = stat \lim_{n\to\infty} x_n$ is a bounded with $\|g\| = 1$.

Proof:

Let $x \in st$

$$|g(x)| = \left|stat \lim_{n\to\infty} x_n\right|$$
$$\leq \left|\sup_{n\in\mathbb{N}} x_n\right|$$
$$\leq \sup_{n\in\mathbb{N}} |x_n|$$
$$= \|x\|_\infty$$
$$\Longrightarrow \frac{|g(x)|}{\|x\|_\infty} \leq 1.$$
$$\Longrightarrow \|g\| \leq 1.$$

Again, consider $y = (\lambda, \lambda, \lambda, \dots) \in st$.

Then $g(y) = \lambda = \|y\|_\infty$.

Thus $\exists\, y \in st$ such that $\frac{|g(x)|}{\|y\|_\infty} = 1 \Longrightarrow \|g\| > 1$.

Hence $\|g\| = 1$.

Thus, by Hahn-Banach Theorem g can be extended to l_∞ preserving norm

that is $\exists\; L \in (l_\infty)^*$ such that $L|_{st} = g$ and $\|L\| = \|g\|$,

where $(l_\infty)^*$ is the continuous dual (dual space) of l_∞.

We now state and prove the main result of the paper. In fact we prove

Existence of Banach Statistical Limit Functional

Theorem 3.2

There exists a functional $\mathcal{F} : l_\infty \to \mathbb{R}$ is named as Banach statistical limit functional satisfying the following :

(i) $\|\mathcal{F}\| = 1$ and $st_0 \subset \ker \mathcal{F}$

(ii) $\mathcal{F}|_{st} = g$, where g is the statistical limit functional on st,

(iii) If $s \in l_\infty$ with$s_n \geq 0$, then $\mathcal{F}(s) \geq 0$,

(iv) If $x \in l_\infty$, then $\mathcal{F}(x) = \mathcal{F}(Tx)$,

where $T :\; l_\infty \to l_\infty$ is a map with $(Tx)_k = (Sx)_k\; a.a.k$ for each $x \in l_\infty$ and S is the shift operator defined by $S((x_n)_{n=1}^\infty) = (x_n)_{n=2}^\infty$.

Proof:

(i) Let $G = \{x - Sx : x \in l_\infty$ and $N = \{y \in l_\infty : y_k = x_k\; a.a.k, x \in\; G\}$.

So $G \subset N \subset l_\infty$.

Clearly G is a subspace of l_∞.

Let $p, q \in N$ and any $\mu, \lambda \in \mathbb{R}$.

Then $\exists\, x, y \in l_\infty$ such that $p_n = (x - Sx)_n\; a.a.n$ and $q_n = (y - Sy)_n\; a.a.n$.

Since, G is a subspace, $\mu(x - Sx) + \lambda(y - Sy) \in G$.

So by Lemma 2.1, we have

$$(\mu p + \lambda q)_n$$
$$= \big(\mu(x - Sx) + \lambda(y - Sy)\big)_n a.a.n$$
$$\Longrightarrow \mu p + \lambda q \in N.$$

Therefore N is a subspace of l_∞.

Now, we claim that $d(1, N) = 1$,

where $1 = (1,1,1, \dots)$.

For, $0 \in G \Longrightarrow d(1, G) \leq 1$.

Since, $G \subset N, d(1, G) \leq 1$.

Let $p \in N$.

Then $p_n = (b - Sb)_n\; a.a.n$ for some $b \in l_\infty$.

If $\quad p_n \leq 0 \quad$ for some $n \in \mathbb{N}$

then $\|1 - p_n\|_\infty = \sup_{n\in\mathbb{N}} |1 - p_n| \geq 1$.

Again, if $p_n \geq 0, \forall\, n \in \mathbb{N}$

then $(b - Sb)_n \geq 0\; a.a.n$

$\Longrightarrow b_n \geq b_{n+1}\; a.a.n.$

Thus $(b_n)_n$ has a subspace $\left(b_{n_k}\right)_k$ such that $b_{n_k} \geq b_{n_{k+1}} \forall\, k \in \mathbb{N}$ with $\delta(\{n_k : k \in \mathbb{N}\}) = 1$.

Since $\left(b_{n_k}\right)_k$ is monotonically decreasing bounded sequence,

$\Rightarrow \lim_{n\to\infty} b_{n_k} = l$ (say) exists.

$\Rightarrow stat \lim_{n\to\infty} b_{n_k} = l.$ by Lemma 2.2

$\Rightarrow stat \lim_{n\to\infty} (b_n - b_{n+1}) = 0$ by Lemma 2.4

$\Rightarrow stat \lim_{n\to\infty} b_n = l$

$\Rightarrow stat \lim_{n\to\infty} p_n = 0$ by Lemma 2.5

Hence using Lemma 2.2, there is a subsequence $\left(p_{n_j}\right)_j$ of $(p_n)_n$

Such that $\delta(\{n_j : j \in \mathbb{N}\}) = 1$ with $\lim_{j\to\infty} p_{n_j} = 0$

So $\|1-p\|_\infty = \sup_{n\in\mathbb{N}} |1-p_n| \geq \sup_{j\in\mathbb{N}} \left|1-p_{n_j}\right| = 1$

Thus, for any $p \in N$, we have

$\Rightarrow \|1-p\|_\infty \geq 1$

$\Rightarrow d(1,N) \geq 1$

Hence $d(1,N) = 1$

Clearly. $1 \notin \bar{N}$.

By the Proposition 2.1 there exists a functional $\mathcal{F} : l_\infty \to \mathbb{R}$

such that $\mathcal{F}(1) = 1, \mathcal{F}(0) = 0, \forall\, y \in N$ with

$\|\mathcal{F}\| = d(1,N)^{-1} = 1$ (a)

Let st_0 be the collection of all bounded sequences which converge statistically to 0.

Claim : we have to show that $st_0 \subset \ker \mathcal{F}$

For, let $\xi \in st_0$ and $\epsilon > 0$ be any real number.

Then $\xi \in l_\infty$ and $\xi_n \xrightarrow{stat} 0$

i.e. if $U_\epsilon = \{k \in \mathbb{N} : |\xi_k| > \epsilon$ then $\delta(U_\epsilon) = 0$

Again $\mathcal{F}(y) = 0, \forall\, y \in N$

$\Rightarrow \mathcal{F}(y) = 0, y_k = (z - Sz)_k\ a.a.k,\ \forall\, z \in G$

So by the linearity of $\mathcal{F}$ we can write $\mathcal{F}(x) = \mathcal{F}(Tx)$

where $T :\ l_\infty \to l_\infty$ is any map

such that $\sigma_T(z) = \{j \in \mathbb{N} : (Tz)_j \neq (Sz)_j\}$ and

$\delta[\sigma_T(z)] = 0, \quad \forall\, z \in l_\infty$

Choosing $\sigma_T(x) = U_\epsilon\ \forall\, x \in l_\infty$,

we consider the map $T : l_\infty \to l_\infty$ defined by

$Tx = r, \forall\, x \in l_\infty$,

where $r_k = \begin{cases} 0, & if\ k+1 \in U_\epsilon \\ x_{k+1}, & otherwise \end{cases}$

Then $\mathcal{F}(\xi) = \mathcal{F}(T\xi)$

$\Rightarrow |\mathcal{F}(\xi)| = |\mathcal{F}(T\xi)| \leq \|\mathcal{F}\|\|T\xi\|_\infty$

$= \sup\{|(T\xi)_k| : k \in \mathbb{N} \leq \epsilon$

Since $\epsilon > 0$ is arbitrary, $\mathcal{F}(\xi) = 0$

Thus $st_0 \subset \ker \mathcal{F}$ (b)

Hence the Theorem 3.2 (i)

(ii) Next we claim that, $\mathcal{F}|_{st} = g$

For, let $x \in st$ i.e. $x_n \xrightarrow{stat} \Omega$ (say)

Then the sequence e defined by

$$e_j = x_j - \Omega, \qquad \forall\, j \in \mathbb{N}$$

Now $x - \Omega 1 = e \in st_0$ by Lemma 2.3

$\Rightarrow e \in \ker \mathcal{F}$

$$\begin{aligned} \text{Now } \mathcal{F}(x) &= \mathcal{F}(x - \Omega 1) + \mathcal{F}(\Omega 1) \\ &= \mathcal{F}(e) + \Omega + \mathcal{F}(1) \\ &= \Omega\mathcal{F}(1) \\ &= \Omega \\ &= stat \lim_{n\to\infty} x_n \\ &= g(x) \end{aligned}$$

Thus $\mathcal{F}|_{st} = g$

Hence proved.

Claim (iii) If $s \in l_\infty$ with$s_n \geq 0$, then $\mathcal{F}(s) \geq 0$.

Proof :

If $u, v \in l_\infty$ with $u_k = v_k\ a.a.k$,

then $\mathcal{F}(u) = \mathcal{F}(v)$

For, let $\omega_n = u_n - v_n,\ \forall\, n \in \mathbb{N}$

Let $K = \{k : \omega_k \neq 0\}$

Then $\delta(K) = 0$

Now consider the map $T :\ l_\infty \to l_\infty$ with $\sigma_T(\omega) = K$ defined by $Ta = b,\ \forall\, a \in l_\infty$,

where $b_k = \begin{cases} 0,\ if\ k+1 \in K \\ a_{k+1},\ otherwise \end{cases}$

Therefore $T\omega = 0$

$$\begin{aligned} \text{Now, } \mathcal{F}(u) - \mathcal{F}(v) &= \mathcal{F}(u - v) \\ &= \mathcal{F}(\omega) \\ &= \mathcal{F}(T\omega) \\ &= \mathcal{F}(0) = 0 \end{aligned}$$

Thus $\mathcal{F}(u) = \mathcal{F}(v)$

If possible, suppose that there exists $z \in l_\infty$ with $z_n \geq 0$ a.a.n

Such that $\mathcal{F}(z) < 0$

Consider $y \in l_\infty$ defined by $y_n = \frac{z_n}{\|z\|_\infty}$

Clearly $y_n \geq 0\ \ a.a.n$ and $\mathcal{F}(y) < 0$

Again consider $x \in l_\infty$ defined by

$$x_k = \begin{cases} 0, & if\ y_k < 0 \\ y_k, & if\ y_k \geq 0 \end{cases}$$

Then, $x_n = y_n,\ a.a.n$ and $0 \leq x_n \leq 1,\ \forall\, n \in \mathbb{N}$

Therefore $\mathcal{F}(x) = \mathcal{F}(y) < 0$ and

$\|1 - x\|_\infty = \sup_{n\in\mathbb{N}} |1 - x_n| \leq 1$

Again $\mathcal{F}(1 - x) = \mathcal{F}(1) - \mathcal{F}(x)$

$= 1 - \mathcal{F}(x) > 1$

Thus we get

$1 < |\mathcal{F}(1 - x)| < \|\mathcal{F}\|\|1 - x\|_\infty$

$= \|1 - x\|_\infty \leq 1$

which is a contradiction.

Hence, if $s \in l_\infty$ with $s_k \geq 0\ \ a.a.k$

then $\mathcal{F}(s) \geq 0$

This completes the proof.

The proof of the next part is simple.

Corollary 3.1

Every Banach statistical limit functional is a Banach limit functional on l_∞.

Corollary 3.2

Let $\mathcal{F}$ be any Banach statistical limit functional on l_∞.

If $u, v \in l_\infty$ with $u_k = v_k\ a.a.k,$

then $\mathcal{F}(u) = \mathcal{F}(v)$.

Now by using Banach statistical limit functional we introduce a new type of convergence called generalized almost statistical convergence (GAS convergence) defined as follows :

Definition 3.1

Let $x \in l_\infty$.

Then x is said to be generalized almost statistically convergent to λ if $\mathcal{F}(x) = \lambda$ for all Banach statistical limit functionals $\mathcal{F}$

i.e. if $\mathcal{F}(x)$ is invariant (unique) for each Banach statistical limit functionals $\mathcal{F}$ on l_∞.

Let S be the set of all GAS convergent real sequences.

Then the following result is easily obtained from the Theorem 3.2 and Definition 3.1.

Corollary 3.3

Every statistically convergent sequence is GAS convergent with the same limit i.e. $st \subset S$.

The converse is not true (see Example 3.1).

Clearly $st \subsetneq S$.

From the Definition 3.1 it follows that every bounded statistically convergent sequence is GAS convergent with the same limit.

Lemma 3.1

Every almost convergent real sequence in GAS convergent with the same limit i.e. $\mathcal{A} \subset S$.

Proof:

Suppose $r = (r_i)_i \in l_\infty$ is an almost convergent real sequence with limit k.

Then for any Banach limit functional $B : l_\infty \to \mathbb{R}$ we have $B(r) = k$.

We can easily say that $\mathcal{F}(r) = k$

for any Banach statistical limit functional $\mathcal{F}$ by Corollary 3.1

Thus $(r_n)_n$ is generalized almost statistically convergent to k.

The converse is not true.

Because the sequence $x = (x_n)_n$ is not almost convergent.

But it is statistically convergent.

$\Rightarrow x \in S$

$\Rightarrow \mathcal{A} \subsetneq S$

Example 3.1

Consider the real bounded sequence $x = (x_n)_n$ defined by

$$x_k = \begin{cases} 5 & \text{if } k \text{ is a perfect square number} \\ 0 & \text{if } k \text{ is even and not a perfect square} \\ 1 & \text{if } k \text{ is odd and not a perfect square} \end{cases}$$

i.e.

$(x_n)_n$
$= (5,0,1,5,1,0,1,0,5,0,1,0,1,0,1,5,1,0,1,0,1,0,1,0,5,0, \ldots)$

Clearly x is not convergent in usual sense.

Even x is not convergent statistically.

But x is GAS convergent to $\frac{1}{2}$.

For let $S = \{n^2 : n \in \mathbb{N}\}$.

Now consider the map $T : l_\infty \to l_\infty$ with $\sigma_T(x) = S$ defined by $Tz = y, \forall\, z \in l_\infty$, where

$$y_k = \begin{cases} z_{k+1} & \text{if } k+1 \notin S \\ 0 & \text{if } k+1 \in S \text{ and } k \text{ is odd} \\ 1 & \text{if } k+1 \in S \text{ and } k \text{ is even} \end{cases}$$

Then $Tx = (0,1,0,1,0,1,\dots)$.

Let $\mathcal{F}$ be any Banach statistical limit functional.

Since $u, v \in l_\infty$ with $u_k = v_k \; a.a.k$

implies $\mathcal{F}(u) = \mathcal{F}(v)$

we have $\mathcal{F}(x) = \mathcal{F}(1,0,1,0,1,0,\dots)$.

Now

$$\begin{aligned} \mathcal{F}(x) &= \mathcal{F}(Tx) \\ &= \mathcal{F}(1,0,1,0,1,0,\dots) \\ &= \mathcal{F}\big((1,1,1,1,1,1,\dots) - (1,0,1,0,1,0,\dots)\big) \\ &= \mathcal{F}(1,1,1,1,1,1,\dots) - (1,0,1,0,1,0,\dots) \\ &= 1 - \mathcal{F}(x) \end{aligned}$$

$$\Rightarrow \mathcal{F}(x) = \frac{1}{2}$$

The interesting example shows that there exists a GAS convergent sequence which is neither almost convergent nor statistically convergent.

i.e. $\boldsymbol{st} \cup \boldsymbol{\mathcal{A}} \subsetneq \boldsymbol{S}$.

Example 3.2

Consider the sequence $\xi = (\xi_n)_n$ defined as follows

$$\xi = \Big(\underbrace{1,0,1,0,\dots}_{100 \text{ terms}}, \overbrace{1,1,1,\dots,1}^{10 \text{ terms}}, \underbrace{1,0,1,0,\dots}_{100^2 \text{ terms}}, \overbrace{1,1,1,\dots,1}^{10^2 \text{ terms}}, \underbrace{1,0,1,0,\dots}_{100^3 \text{ terms}}, \overbrace{1,1,\dots,1}^{10^3 \text{ terms}} \dots\Big)$$

It is easy that ξ is neither a statistically convergent nor an almost convergent sequence.

But ξ is GAS convergent.

For let $E = \bigcup_{i=1}^{\infty} \big(b_i - 10^i, b_i\big] \cap 2\mathbb{N}$ with $b_i = \sum_{j=1}^{i} \big(100^j + 10^j\big)$ for each $i \in \mathbb{N}$.

Let $T : l_\infty \to l_\infty$ defined by $Tx = z$ with

$$z_k = \begin{cases} x_{k+1}, & \text{if } k+1 \notin E \\ 0, & \text{otherwise} \end{cases}$$

Then $z = (0,1,0,1,0,\dots)$.

Since $\delta(E) = 0, z_k = (Sx)_k \; a.a.k$.

Therefore

$$\begin{aligned} \mathcal{F}(x) &= \mathcal{F}(Tx) \\ &= \mathcal{F}(0,1,0,1,0,\dots) \\ &= \mathcal{F}(1,0,1,0,\dots) \\ &= \frac{1}{2} \end{aligned}$$

Theorem 3.3

GAS convergence cannot be characterized by ideal convergence for proper ideals of $\mathbb{N}$

Proof.

If possible suppose that GAS convergence coincides with ideal convergence for some proper ideal I of $\mathbb{N}$.

Since $\xi = (1,0,1,0,\dots.)$ is almost convergent to $\frac{1}{2}$

$\therefore \; \xi$ is GAS convergent to $\frac{1}{2}$.

Hence For any $\epsilon > 0$,

$$A_\epsilon = \left\{k \in \mathbb{N} : \left|\xi_k - \tfrac{1}{2}\right| \geq \epsilon\right\} \in I$$

But $A_{\frac{1}{2}} = \mathbb{N} \in I$

which contradicts that I is a proper ideal of $\mathbb{N}$.

Hence the theorem.

Topological properties of the space of all GAS convergent sequences

Lemma 3.2

Some topological properties of space $S = \{s^{(n)} \,|\, n \in \mathbb{N}\}$ of all GAS convergent sequences.

(i) S is closed

(ii) S is non-separable in l_∞

(iii) S is first countable but not second countable.

(iv) S is not Lindelof and not compact.

Proof:

(i) S is closed. For let $s \in S$.

Then by the sequence lemma, there exists a sequence $\big(s^{(n)}\big)_{n\in\mathbb{N}}$ in S such that $\lim_{n\to\infty} s^{(n)} = s$.

Let ρ, τ be any two Banach statistical limit functional.

Since ρ, τ are continuous

$$\begin{aligned} \rho(s) &= \rho\left(\lim_{n\to\infty} s^{(n)}\right) \\ &= \lim_{n\to\infty}\left(\rho\big(s^{(n)}\big)\right) \\ &= \lim_{n\to\infty}\left(\tau\big(s^{(n)}\big)\right) \\ &= \tau\left(\lim_{n\to\infty} s^{(n)}\right) \\ &= \tau(s). \end{aligned}$$

Hence S is closed.

(ii) S is non-separable.

For consider $\Lambda \subset l_\infty$ defined as follows

$$\Lambda = \left\{ x \to l_\infty : x_k = \begin{cases} 0 \; or \; 1 & if \; k \; is \; a \; perfect \; square \\ 0 & otherwise \end{cases} \right\}$$

Clearly Λ is an uncountable subset of $\mathcal{A} \subset S$ which implies that S is uncountable.

Now for any distinct $u, v \in \Lambda, d(u, v) = \|u - v\|_\infty = 1$.

Thus Λ is an uncountable discrete subset of S.

Let D be any dense set in S i.e. $\overline{D} = S$.

Let us consider any $s, t \in \Lambda$ with $s \neq t$.

Then $s, t \in S = \overline{D}$.

Then the disjoint open balls $B_d\left(s, \frac{1}{2}\right)$ and $B_d\left(t, \frac{1}{2}\right)$ must have non-empty intersections with D which implies that there are two distinct elements of D.

Since Λ is uncountable

So D is also uncountable.

Thus S does not contain any countable dense subset i.e. S is non-separable in l_∞.

Proof of (iii) and (iv) are very easy since

(iii) S is not second countable due to its non-separability.

(iv) Since S is metrizable and not second countable.

So S is non-Lindelof and non-compact.

This completes the proof.

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