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Generalised Limits for Vector Valued Sequences

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ABSTRACT

The object of the present paper is to introduce the extension of the concept of Banach limits for vector valued sequences and we prove the existence of Banach limits for vector valued sequences. Also introduced Banach spaces X, X^*, X^{**} 1-complemented in their bi duals admit vector valued Banach limits. Lastly we propose Lorentz's vector valued intrinsic characterisation of almost convergence.

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Keywords : Banach limit, Almost convergence , Vector valued Banach limits, Vector valued Intrinsic Characterisation of Almost Convergence

I. INTRODUCTION & PRELIMINARIES

The existence of Banach limit functionals was proven by Banach (see [6]) in 1932. Using Banach limits, in 1948, Lorentz (see [4]) introduced the notion of almost convergence, which is a generalization of usual convergence of real sequences. Again in 1932 Theorie des operations lineaires (see [6]) Banach extended, in a natural way, operation of taking limit defined on the space of all convergent real sequences to the space of all bounded real sequences. Lorentz proposed a bounded sequence $(x_n)_{n\in\mathbb{N}} \in l_\infty$ is called almost convergent if there exists a number $y \in \mathbb{R}$ (called the almost limit of $(x_n)_{n\in\mathbb{N}}$) such that $\varphi((x_n)_{n\in\mathbb{N}}) = y$ for all Banach limits $\varphi : l_\infty \to \mathbb{R}$.

His criterion for almost convergence that a bounded sequence $(x_n)_{n\in\mathbb{N}} \in l_\infty$ is almost convergent to a real number y if and only if

$\lim_{p\to\infty} \frac{1}{p+1} \sum_{k=0}^{p} x_{n+k} = y$ uniformly in $n \in \mathbb{N}$

This criterion is known as **Lorentz intrinsic characterisation of almost convergence**.

In (see [5]) intrinsic characterisation as a model for extending the concept of almost convergence to vector valued sequences.

A sequence $(x_n)_{n\in\mathbb{N}}$ is a real normed space X is called almost convergence, if there exists $y \in X$ (called the almost limit of $(x_n)_{n\in\mathbb{N}}$ and denoted by $AC \lim_{l\to\infty} x_n = y$) such that $\lim_{p\to\infty} \frac{1}{p+1} \sum_{k=0}^{p} x_{n+k} = y$ uniformly in $n \in \mathbb{N}$.

In (see [5]) every convergent sequence must be bounded.

In fact, given a real normed space X, the set $ac(X)$ of all almost convergent sequences in X is a closed subspace of $l_\infty(X)$.

On the other hand, the operation of taking almost limit, which is defined as

$AC \lim : ac(X) \to X$,

$(x_n)_{n\in\mathbb{N}} \to AC \lim_{l\to\infty} x_n$

is a norm – 1 continuous linear operator (see [1], [5]).

Next we present a vector valued version of Lorentz almost convergence intrinsic characterisation.

The scope of this paper is

Section 2 : The existence of Banach limit for vector valued sequences.

Section 3 : Investigation of Lorentz intrinsic characterisation of almost convergence.

II. THE EXISTENCE OF BANACH LIMITS FOR VECTOR VALUED SEQUENCES

Definition 1 :

A linear function $\varphi : l_\infty \to \mathbb{R}$ is called a Banach limit if

(i) $\varphi((x_n)_{n\in\mathbb{N}}) \geq 0$ for $(x_n)_{n\in\mathbb{N}} \in l_\infty$ such that $x_n \geq 0,\ \forall\, n \in \mathbb{N}$

(ii) $\varphi((x_n)_{n\in\mathbb{N}}) = \varphi((x_{n+1})_{n\in\mathbb{N}})$ for $(x_n)_{n\in\mathbb{N}} \in l_\infty$

(iii) $\varphi(1) = 1$, where 1 denotes the constant sequence of ones.

A linear function $\varphi : l_\infty \to \mathbb{R}$ is a Banach limit if and only if φ is invariant under the shift operator on l_∞ and $\lim \inf_{n\to\infty} x_n \leq \varphi((x_n)_{n\in\mathbb{N}}) \leq \lim \sup_{n\to\infty} x_n, \quad \forall\, (x_n)_{n\in\mathbb{N}} \in l_\infty$

Every Banach limit $\varphi : l_\infty \to \mathbb{R}$ satisfies the conditions

(i) $\varphi|_c = \lim$

(ii) $\|\varphi\| = 1$

Therefore, Banach limits are norm − 1 Hahn-Banach extensions of the limit operation from c to l_∞.

On the other hand, in 2010 Semenov and Sukochev (see [3]) studied invariant Banach limits and proved the existence of a Banach limit B such that $B = B\ o\ H$ for all bounded linear operators H on l_∞ satisfying easily verifiable conditions.

The concept of Banach limit to the vector valued case were made in (see [2], [7]). In order to extend the concept of Banach limit to vector valued sequence it suffices to notice that if $\psi \in S_{l_\infty^*}$ (by S with a subscript we denote the unit sphere in the corresponding space) and $\psi(1) = 1$, then ψ satisfies condition (i) in Definition 1 : if $(x_n)_{n\in\mathbb{N}} \in S_{l_\infty}$ and $x_n \geq 0,\ \forall\, n \in \mathbb{N}$, then $(1 - x_n)_{n\in\mathbb{N}} \in B_{l_\infty}$ (by B with a subscript we denote a ball in the corresponding space).

Hence $1 = \psi(1) = \psi((x_n)_{n\in\mathbb{N}}) + \psi((1 - x_n)_{n\in\mathbb{N}}) \leq \psi((x_n)_{n\in\mathbb{N}}) + 1$

Bearing this is mind, we can define a Banach limit for real bounded sequences as follows : A functional $\varphi : l_\infty \to \mathbb{R}$ is a Banach limit if and only if $\varphi \in S_{l_\infty^*}$, φ is invariant under the shift operator on l_∞ and $\varphi|_c = \lim$

Now, define Banach limits for vector-valued sequences.

Definition 2:

Let X be a real normed space.

The set of vector valued Banach limits on X is defined as $\mathcal{B}\quad \mathcal{L}(X) = S_{\mathcal{CL}(l_\infty(X),X)} \cap \mathcal{N}_X \cap \mathcal{L}_X$ where $\mathcal{CL}(l_\infty(X), X)$ denotes the space of all continuous linear operators $T : l_\infty(X) \to X$ and $\mathcal{N}_X$ denotes the set of all continuous linear operators $T : l_\infty(X) \to X$ which are invariant under the shift operator on $l_\infty(X)$ and $\mathcal{L}_X$ denotes the set of all continuous linear operators $T : l_\infty(X) \to X$ which are extensions of the limit operation on $c(X)$.

Theorem 1:

Let X^* be a real dual Banach space.

Then there exists a Banach limit $\varphi \in \mathcal{BL}(X^*)$ such that if $(x_n^*)_{n\in\mathbb{N}} \in l_\infty(X^*)$ is ω^* - Cesaro convergent to $x^* \in X^*$, then $\omega((x_n^*)_{n\in\mathbb{N}}) = x^*$

proof :

Consider the filter on $\mathbb{N}$ consisting of the sets $\{A \subseteq \mathbb{N} : \mathbb{N}\backslash A \text{ is finite}\}$.

There exists a functional $\mathcal{F}$ on $\mathbb{N}$ containing this filter. Obviously, $\mathcal{F}$ is not of the form

$\mathcal{U}_n = \{A \subseteq \mathbb{N} : n \in A\}$, for any $n \in \mathbb{N}$

We set $\varphi : l_\infty(X^*) \to X^*$,

$$(x_n^*)_{n\in\mathbb{N}} \to \mathcal{F} - \lim \frac{x_1^* + \cdots + x_n^*}{n}$$

(1) The map φ is well defined.

Indeed, note that $B_{X^*}(0, \|(x_n^*)_{n\in\mathbb{N}}\|_\infty)$ endowed with the ω^* - topology compact and Hausdoff. Since, every function on a compact Hausdoff topological space is convergent to a unique limit, we deduce that $\mathcal{F}-\lim\frac{x_1^*+\cdots+x_n^*}{n}$ exists and belongs to $B_{X^*}(0,\|(x_n^*)_{n\in\mathbb{N}}\|_\infty)$.

(2) The map φ is linear and continuous and has norm 1. Indeed, it follows from assertion (1) above that $\|\varphi\|\le 1$. In order to see that $\|\varphi\|=1$, it suffices to consider any constant sequence of norm 1.

(3) The condition $\varphi|_{c(X^*)}=\lim$ holds. Indeed, if $(x_n^*)_{n\in\mathbb{N}}$ is a sequence in X^* converging to $x^*\in X^*$, then $\left(\frac{x_1^*+\cdots+x_n^*}{n}\right)_{n\in\mathbb{N}}$ converges to x^*.

Therefore, $\left(\frac{x_1^*+\cdots+x_n^*}{n}\right)_{n\in\mathbb{N}}$ ω^* - converges to x^*.

Hence, $\mathcal{F}-\lim\frac{x_1^*+\cdots+x_n^*}{n}=x^*$ by the construction of the ultrafilter $\mathcal{F}$.

(4) The map φ is invariant under the shift operator. Indeed, for every $(x_n^*)_{n\in\mathbb{N}}\in l_\infty(X^*)$ we have

$$\varphi((x_{n+1}^*-x_n^*)_{n\in\mathbb{N}})=\mathcal{F}-\lim\frac{x_1^*+\cdots+x_n^*}{n}=0$$

Since, $\left(\frac{x_1^*+\cdots+x_n^*}{n}\right)_{n\in\mathbb{N}}$ converges to 0.

(5) If $(x_n^*)_{n\in\mathbb{N}}\in l_\infty(X^*)$ is ω^* - Cesaro convergent to $x^*\in X^*$, then $\varphi((x_n^*)_{n\in\mathbb{N}})$. Indeed, $\left(\frac{x_1^*+\cdots+x_n^*}{n}\right)_{n\in\mathbb{N}}$ is ω^* - convergent to x^*, whence

$$\mathcal{F}-\lim\frac{x_1^*+\cdots+x_n^*}{n}=x^*$$

Note :

If X is a real Banach space, $\varphi\in\mathcal{BL}(X)$ and Y is a $1-$ complimented subspace of X, then $l_\infty(Y)\to Y$, $(y_n)_{n\in\mathbb{N}}\to p\big(\varphi((y_n)_{n\in\mathbb{N}})\big)$ is a Banach limit on Y, where $p:X\to Y$ is a norm -1 projection.

Thus, we have the following corollary

Corollary 1 :

Let X be a real Banach space.

If X is a $1-$ complemented in X^{**}

then $\mathcal{BL}(X)\neq\phi$.

This is well known.

Example: A Banach space free of vector valued Banach limits is $c_0(\mathcal{BL}(c_0)=\phi)$

Indeed if $\varphi\in\mathcal{BL}(c_0)$

then the map $p:l_\infty\to c_0$,

$(y_n)_{n\in\mathbb{N}}$

$\to\varphi\big((y_1,0,0,\dots),(y_2,y_2,0,0,\dots),(y_3,y_3,y_3,0,0,\dots),\dots\big)$

is a bounded linear projection, which can not exists.

Since, c_0 is not complemented in l_∞

III. Lorentz' vector valued intrinsic characterisation of almost convergence

A bounded sequence $(x_n)_{n\in\mathbb{N}}\in l_\infty$ is called almost convergent if there exists a number $y\in\mathbb{R}$ (called the almost limit of $(x_n)_{n\in\mathbb{N}}$) such that $\varphi((x_n)_{n\in\mathbb{N}})=y$ for all Banach limits $\varphi:l_\infty\to\mathbb{R}$.

Lorentz proposed the following criterion for almost convergence : A bounded sequence $(x_n)_{n\in\mathbb{N}}\in l_\infty$ is almost convergent to a real number y if and only if

$\lim\limits_{p\to\infty}\frac{1}{p+1}\sum_{k=0}^{p}x_{n+k}=y$ uniformly in $n\in\mathbb{N}$

This criterion is known as Lorentz intrinsic characterisation of almost convergence.

In (see [5]) intrinsic characterisation as a model for extending the concept of almost convergence to vector valued sequences.

A sequence $(x_n)_{n\in\mathbb{N}}$ is a real normed space X is called almost convergence, if there exists a $y\in X$ (called the almost limit of $(x_n)_{n\in\mathbb{N}}$ and denoted by $AC\lim\limits_{l\to\infty}x_n=y$) such that $\lim\limits_{p\to\infty}\frac{1}{p+1}\sum_{k=0}^{p}x_{n+k}=y$ uniformly in $n\in\mathbb{N}$.

In (see [5]) every convergent sequence must be bounded.

In fact, given a real normed space X, the set $ac(X)$ of all almost convergent sequences in X is a closed subspace of $l_\infty(X)$.

On the other hand, the operation of taking almost limit, which is defined as

$AC\lim:\ ac(X)\to X,$

$(x_n)_{n\in\mathbb{N}}\to AC\lim\limits_{l\to\infty}x_n$ (1)

is a norm -1 continuous linear operator (see [1], [5]).

Next, we represent a vector valued version of Lorentz intrinsic characterisation of almost convergence.

In fact we prove

.Theorem 2:

Let X be a real normed space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X.

Then the following conditions are equivalent :

(A) $(x_n)_{n\in\mathbb{N}}$ is almost convergent to 0 in X

(B) $T((x_n)_{n\in\mathbb{N}}) = 0, \ \forall\, T \in \mathcal{N}_X$

Proof (A) :

Assume that $(x_n)_{n\in\mathbb{N}}$ is almost convergent to 0 in X

Fix $T \in \mathcal{N}_X\{0\}$.

We set $s = T((x_n)_{n\in\mathbb{N}})$.

Let us show that $\|s\| < \epsilon, \ \forall\, \epsilon > 0$.

Taking an arbitrary $\epsilon > 0$

There exists a $p \in \mathbb{N}$ such that

$$\left\|\frac{1}{p+1}\sum_{k=0}^{p} x_{n+k}\right\| < \frac{\epsilon}{\|T\|}, \qquad \forall\ n \in \mathbb{N}$$

We observe that

$$s = T(x_1, x_2, x_3, x_4, x_5, \dots)$$
$$s = T(x_2, x_3, x_4, x_5, x_6, \dots)$$
$$s = T(x_3, x_4, x_5, x_6, x_7, \dots)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$s = T\left(x_{p+1}, x_{p+2}, x_{p+3}, x_{p+4}, x_{p+5}, \dots\right)$$

Therefore,

$$(p+1)s = T\left(x_1 + \cdots + x_{p+1}, x_2 + \cdots + x_{p+2}, \dots\right)$$
$$\Rightarrow s = T\left(\frac{x_1 + \cdots + x_{p+1}}{p+1}, \frac{x_2 + \cdots + x_{p+2}}{p+1}, \dots\right)$$
$$\Rightarrow \|s\| = \left\|T\left(\frac{x_1 + \cdots + x_{p+1}}{p+1}, \frac{x_2 + \cdots + x_{p+2}}{p+1}, \dots\right)\right\|$$
$$\le \|T\| \left\|\left(\frac{1}{p+1}\sum_{k=0}^{p} x_{n+k}\right)_{n\in\mathbb{N}}\right\|_\infty$$
$$< \epsilon$$

Proof (B):

Assume that $T((x_n)_{n\in\mathbb{N}}) = 0, \ \forall\, T \in \mathcal{N}_x$ and suppose that $(x_n)_{n\in\mathbb{N}}$ is not almost convergent to 0. $bps(X)$ (the set of sequences in X with bounded partial sums) coincides with the set

$$\{(z_{n+1} - z_n)_{n\in\mathbb{N}} : (z_n)_{n\in\mathbb{N}} \in l_\infty(X)\}$$

On the other hand, all sequences of the form $(z_{n+1} - z_n)_{n\in\mathbb{N}}$, where $(z_n)_{n\in\mathbb{N}} \in l_\infty(X)$, are almost convergent to 0.

Since, the almost limit operation (see eq(1)) is continuous and $ac(X)$ is closed in $l_\infty(X)$, we deduce that the space of all sequences almost convergent to 0 is also closed in $l_\infty(X)$.

Now, consider the continuous linear map

$$S : bps(X) \oplus \mathbb{R}(x_n)_{n\in\mathbb{N}} \to \mathbb{R}x \subseteq X,$$
$$(z_{n+1} - z_n + \lambda x_n)_{n\in\mathbb{N}} \to \lambda x$$

where $x \in X/\{0\}$ can be chosen arbitrarily.

By the Hahn-Banach theorem we can extend S to a continuous linear operator

$$\hat{S} : l_\infty(X) \to \mathbb{R}x \subseteq X$$

Note that $\hat{S} \in \mathcal{N}_X$

This contradicts to $S((x_n)_{n\in\mathbb{N}}) = x \neq 0$

Hence the Theorem.

Corollary 2:

Let X be a real normed space . If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X almost convergent to $x \in X$, then $T((x_n)_{n\in\mathbb{N}}) = x, \ \forall\ T \in \mathcal{N}_X \cap \mathcal{L}_X$.

In particular, $\varphi((x_n)_{n\in\mathbb{N}}) = x, \ \forall\, \varphi \in \mathcal{BL}(X)$

Corollary 3:

Let X be a real normed space. Then $\mathcal{BL}(X)$ coincides with the set of all norm – 1 Hahn-Banach extensions of AC lim to the whole space $l_\infty(X)$.

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