Density of Function in the Mathematical Analysis

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ABSTRACT

On the ground of the definition of arbitrary function of one argument and the mutual and unambiguous correspondence between the values of the function and argument, the following concepts have been deveined:

a) Absolute average and differential linear density of the function and argument;
b) Relative linear density of one function toward another function;
c) Relative linear density of the argument of one function toward the argument of another function.

The conditions are shown under which these definitions apply: continuity and differentiability of the function; monotony of the function; equal quality of the arrays of values of argument and function and the examination has been done of the specified densities.

The generalization of the concepts of the density of function and argument has been done in the case of a function of two and three variables and the theorem on projections of vector quantity and tensor quantity on the co-ordinate axes and planes has been proven.
Keywords: Density of Function, Monotony of Function.

I. INTRODUCTION

Density of function of one variable

Definition : Let consider an arbitrary function y = f (x) of one variable [1] defined in the interval $[x_1, x_2]$. The graphic of the function is presented on Figure 1.

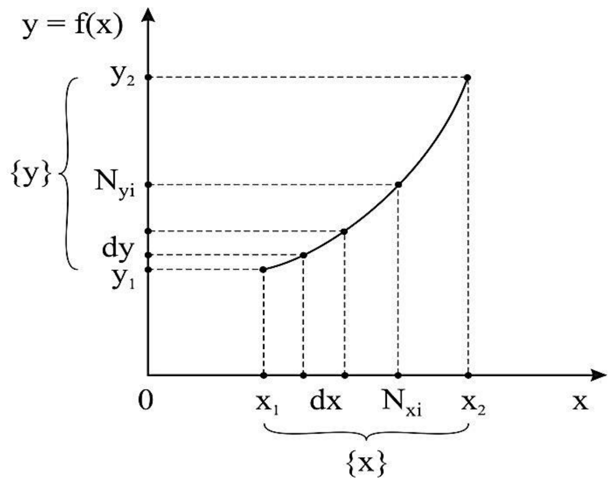


Figure 1. Graphic of the arbitrary function

We will impose the following conditions on this function:

1. f (x) to be continuous and differentiable in this interval.
2. The function to be monotonous in the interval.
3. The arrays of values of the argument and function to have equal quality.

Let denote the array of values of the argument by {x}. This array is continuous with power of continuum (contains all real numbers in the interval $[x_1, x_2]$). The number of elements of it is infinitely large: $N_{xi} \to \infty$. Corresponding array of values of the function {y}, is also continuous, with power of continuum in the interval $[y_1, y_2]$ and number of elements: $N_{yi} \to \infty$. According to the definition of the function and condition 2) between the elements of {x} and {y} exists mutual, unique and reversible correspondence: $N_{xi} \leftrightarrow N_{yi}$, i.e. on each element of the one array corresponds one well-determined element of the other array and vice versa.

1. Absolute Average Densities

Let define the concept of absolute average linear density of the array of values of the argument as:

$$\bar{\rho}_x = \frac{\{N_{xi}\}}{x_2 - x_1} = \frac{\{N_{xi}\}}{\Delta x}$$

(It is clear that according to this definition $\bar{\rho}_x \to \infty$)

We shall accordingly define the concept of absolute average linear density of the array of values of the function (the average density of the function) as:

$$\bar{\rho}_y = \frac{\{N_{yi}\}}{|y_2 - y_1|} = \frac{\{N_{yi}\}}{|\Delta y|}$$

as here $\bar{\rho}_y \to \infty$ as well.

Of the mutual and unambiguous correlation between the values of {x} and {y} follows that:

$\{N_{yi}\} = \{N_{xi}\}$

Then above:

$$\bar{\rho}_y |\Delta y| = \{N_{yi}\} = \{N_{xi}\} = \bar{\rho}_x \Delta x$$

Or:

$$\frac{\bar{\rho}_y}{\bar{\rho}_x} = \frac{\Delta x}{|\Delta y|} \gtreqless 1$$

It is seen that the density of the function may be equal to or different from the density of the argument.

2. Absolute differential densities

The same definition applies to the concepts of absolute differential linear density of the arrays of values of the argument and function when the intervals dx and dy are infinitesimals (fig. 1):

$$\rho_x = \frac{\{dN_{xi}\}}{dx} \quad , \quad \rho_y = \frac{\{dN_{yi}\}}{|dy|}$$

$$\frac{\rho_y}{\rho_x} = \frac{dx}{|dy|} \gtreqless 1$$

3. Relative Density of the Function

Under the relative linear density of one function $y_1 = f_1(x)$ toward another one $y_2 = f_2(x)$ (Figure 2), defined, continuous, differentiable and monotonic in the interval $[x_1, x_2]$ we shall understand respectively:

$$\bar{\rho}_{1y} = \frac{\{N'_{yi}\}}{|y'_2 - y'_1|} = \frac{\{N'_{yi}\}}{|\Delta y'|} \quad , \quad \bar{\rho}_{2y} = \frac{\{N''_{yi}\}}{|y''_2 - y''_1|} = \frac{\{N''_{yi}\}}{|\Delta y''|}$$

$$\frac{\bar{\rho}_{2y}}{\bar{\rho}_{1y}} = \frac{|\Delta y'|}{|\Delta y''|} \gtreqless 1$$

$$\rho_{1y} = \frac{\{dN'_{yi}\}}{|dy'|} \quad , \quad \rho_{2y} = \frac{\{dN''_{yi}\}}{|dy''|}$$

$$\frac{\rho_{2y}}{\rho_{1y}} = \frac{|dy'|}{|dy''|} \gtreqless 1$$

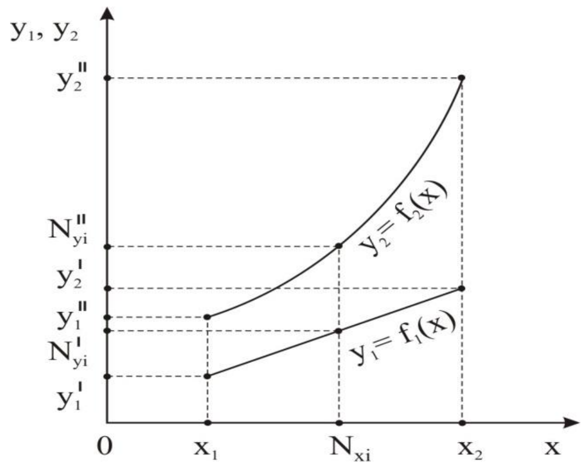


Figure 2. Relative density of the function

4. Relative Density of the Argument

Because of the reversibility of functions we can also consider the relative density of the argument of one function $y_1 = f_1(x')$ toward the argument of another such function $y_2 = f_2(x')$ at the equal density of the arrays of values of functions: $\rho_{1y} = \rho_{2y}$ (fig. 3).

Accordingly, we shall have:

$$\bar{\rho}_{1x} = \frac{\{N'_{xi}\}}{|x'_2 - x'_1|} = \frac{\{N'_{xi}\}}{|\Delta x'|} \quad , \quad \bar{\rho}_{2x} = \frac{\{N''_{xi}\}}{|x''_2 - x''_1|} = \frac{\{N''_{xi}\}}{|\Delta x''|}$$

$$\frac{\bar{\rho}_{2x}}{\bar{\rho}_{1x}} = \frac{|\Delta x'|}{|\Delta x''|} \gtreqless 1$$

The same concerns to differential densities of the arguments as well.

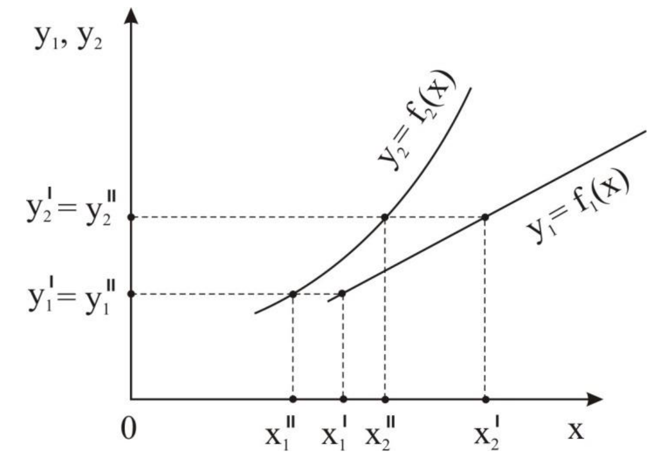


Figure 3. Relative density of the argument

A question remains open with defined in such a manner density of the function and density of the argument: why the density of the corresponding arrays {x} and {y} changes – whether the distance between their values is changed (which contradicts to their continuity with power of continuum) or the values themselves increase their size (which contradicts the fact that they are presented on the number axis as points without any size)?

II. METHODS AND MATERIAL

A. Examination of the density of function

As can be seen from Figure 1, the relationship between ρ_y and ρ_x is determined by the derivative of the function: $|y'| = |df|/dx \in [0, \infty)$:

$$\frac{\rho_y}{\rho_x} = \frac{1}{|y'|} \gtreqless 1$$

When $|y'| > 1$ follows $\rho_y < \rho_x$, when: $|y'| < 1$ - $\rho_y > \rho_x$. Only when $|y'| = 1$ - $\rho_y = \rho_x$. When $|y'| = 0$ ($f(x) = const$) $\rho_y \to \infty$, and when: $|y'| \to \infty$ $\rho_y \to 0$.

We shall consider that the differential density of the array of the argument is equal throughout the whole interval $[x_1, x_2]$. Then for the relationship between the average and differential density we shall have:

$$\bar{\rho}_x = \frac{1}{\Delta x}\int_{x_1}^{x_2}\rho_x dx = \frac{\rho_x}{\Delta x}\int_{x_1}^{x_2} dx = \rho_x$$

$$\bar{\rho}_y = \frac{1}{|\Delta y|}\int_{y_1}^{y_2}\rho_y dy$$

as generally $\rho_y \neq const$.

Here also the relationship between the differential relative densities of two functions is determined by the relationship between their derivatives:

$$\frac{\rho_{2y}}{\rho_{1y}} = \frac{|y_2'|}{|y_1'|} \gtreqless 1$$

Example: (Fig. 4).

Let consider two functions: $y_1 = 2x$ и $y_2 = x^2$, defined in the interval $[x_1=2, x_2]$ with derivatives: $\left(\frac{dy}{dx}\right)_1 = 2 = const$ and $\left(\frac{dy}{dx}\right)_2 = 2x \neq const$. The densities of their arguments are equal: $\rho_x = const$.

$$\bar{\rho}_x = \frac{1}{\Delta x}\int_{x_1}^{x_2}\rho_x dx = \frac{\rho_x}{\Delta x}\int_{x_1}^{x_2} dx = \rho_x = const$$

We also have:

$$\rho_{1y} = \frac{1}{\left(\frac{dy}{dx}\right)_1}\rho_x = \frac{\rho_x}{2} = const$$

$$\rho_{2y} = \frac{1}{\left(\frac{dy}{dx}\right)_2}\rho_x = \frac{\rho_x}{2x} \neq const$$

$$\rho_{2y} < \rho_{1y}$$

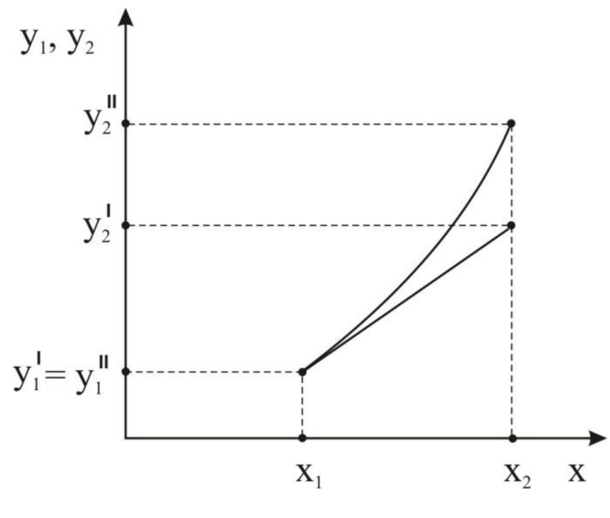


Figure 4. Examination of the density of two functions

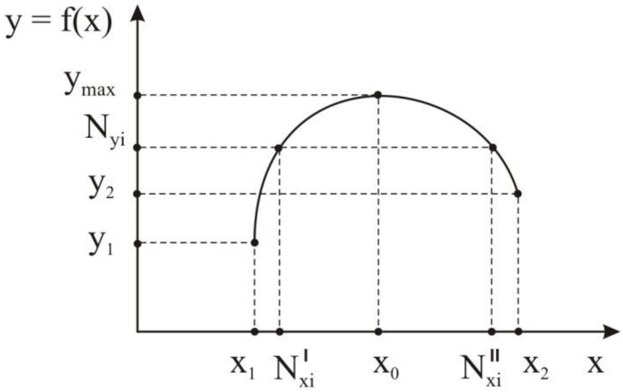
For the average densities of both functions we have respectively:

$$\bar{\rho}_{y1} = \frac{1}{\Delta y}\int_{y_1}^{y_2}\rho_{y1}\,dy = \frac{1}{\Delta y}\int_{y_1}^{y_2}\frac{1}{(dy/dx)_1}\rho_x\,dy = \frac{\rho_x}{\Delta y}\int_{y_1}^{y_2}dx$$

$$= \frac{\rho_x}{\Delta y}x\Big|_{y_1}^{y_2} = \frac{\rho_x}{\Delta y}x\Big|_{x_1}^{x_2} = \rho_x\frac{\Delta x}{\Delta y} = \frac{\rho_x}{2} = \frac{\rho_x}{2} < \rho_x$$

$$\bar{\rho}_{y2} = \frac{1}{\Delta y}\int_{y_1}^{y_2}\rho_{y2}\,dy = \frac{1}{\Delta y}\int_{y_1}^{y_2}\frac{1}{(dy/dx)_2}\rho_x\,dy = \frac{\rho_x}{\Delta y}\int_{y_1}^{y_2}dx$$

$$= \frac{\rho_x}{\Delta y}x\Big|_{y_1}^{y_2} = \frac{\rho_x}{\Delta y}x\Big|_{x_1}^{x_2} = \rho_x\frac{\Delta x}{\Delta y} < \bar{\rho}_{y1} < \rho_x$$



(b)

Figure 5. Density of non-monotony function

B. Extension of the Concepts for the Density of Function and Argument.

We shall try to expand the concepts of density of function and argument as we release from the imposed above conditions.

1. Unlimited intervals of the argument and function

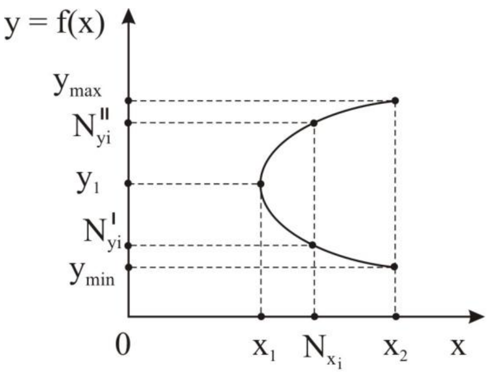
When definitional interval of the argument is unlimited: x € (-∞, + ∞), and range of values of the function is unlimited as well: y € (-∞, +∞), the concepts of differential density of function and argument do not change. The concept of average density of function remains unchanged as well, as now it is possible: $\bar{\rho}_y \in [0, +\infty)$

2. Non-monotony of function.

If the function y = f (x) is not monotonous in the interval **Error! Not a valid link.** then the mutual and unambiguous correspondence between the number of values of argument and function disturbs:

$$\{N_{yi}\} \neq \{N_{xi}\}$$

(on a particular value of x can correspond several values of the function y and vice versa - figure 5 a, b).



(a)

In this case, the concepts of differential density of function and of argument can be generalized considering the fact that they are defined for infinitesimal intervals dx and dy. They remain the same except for the case where $|dy/dx| = 0, +\infty$ (Figure 5 b, a, respectively). There $\rho_y \to \infty$ or $\rho_y \to 0$.

Regarding the concepts of average densities of function and of argument generalization can be done in the following way:

We divide the arrays of values of the function and argument to subarrays where the mutual and unambiguous correspondence is complied: $N_{yi} = N_{xi}$. (For example, for figure 5a these are intervals: $[x_1, x_2]$ and $[y_1, y_{max}]$, respectively: $[x_1, x_2]$ and $[y_1, y_{min}]$; for Figure 5 b. - $[y_1, y_{max}]$ and $[x_1, x_0]$, respectively: $[y_{max}, y_2]$ and $[x_0, x_2]$). For those subarrays concept of average density of function and argument is in force.

Example (Figure 6).

Let consider the function: $y = \sqrt{R^2 - x^2}$ with definitional array of the argument: –R ≤ x ≤ R. The graph of this function is a circle in the plane 0xy with a centre in point 0 and with radius R. The function is non-monotonous in this interval as on the each value of x correspond two values of y and vice versa.

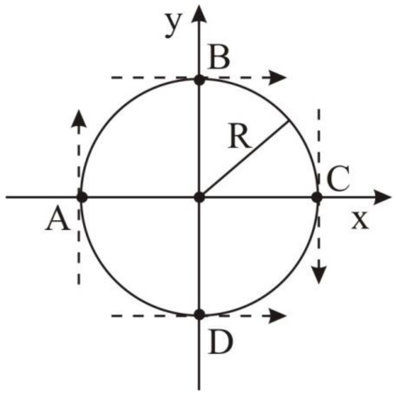


Figure 6. Density of non-monotony function

For the differential densities of the function and argument we have:

$$\rho_x = const$$

$$\rho_y = \rho_y(x) \neq const$$

The derivative of the function is:

$$\frac{dy}{dx} = -\frac{x}{\sqrt{R^2 - x^2}}$$

As at: $x = \pm R,\ \frac{dy}{dx} \to \infty$ and at: $x = 0,\ \frac{dy}{dx} = 0$

Then:

$$\rho_y = \frac{1}{\left|dy/dx\right|}\rho_x = \frac{\sqrt{R^2 - x^2}}{x}\rho_x \neq const$$

In order to find the average densities of the function and the argument we divide arrays {x} and {y} in parts between points (AB), (BC), (AD) and (DC). Then:

$$\bar{\rho}_x = \bar{\rho}_{xA0} = \bar{\rho}_{x0C} = \rho_x$$

For the part (AB):

$$\bar{\rho}_{yAB} = \frac{1}{R}\int_0^R \rho_y dy = \frac{1}{R}\int_0^R \frac{1}{dy/dx}\rho_x dy =$$

$$= \frac{\rho_x}{R}\int_0^R dx = \frac{\rho_x}{R}\int_{-R}^0 dx = \frac{\rho_x}{R}R = \rho_x = \bar{\rho}_{xA0}$$

For the part (BC):

$$\bar{\rho}_{yBC} = \frac{1}{-R}\int_R^0 \rho_y dy = \frac{1}{-R}\int_R^0 \frac{1}{\left|dy/dx\right|}\rho_x dy =$$

$$= \frac{\rho_x}{-R}\int_R^0 -\frac{1}{dy/dx}dx = \frac{\rho_x}{R}\int_0^R dx = \frac{\rho_x}{R}R = \rho_x = \bar{\rho}_{x0C}$$

For the part (AD):

$$\bar{\rho}_{yAD} = \frac{1}{-R}\int_0^{-R} \rho_y dy = \frac{1}{-R}\int_0^{-R} \frac{1}{\left|dy/dx\right|}\rho_x dy =$$

$$= \frac{\rho_x}{-R}\int_0^{-R} -\frac{1}{dy/dx}dx = \frac{\rho_x}{R}\int_{-R}^0 dx = \frac{\rho_x}{R}R = \rho_x = \bar{\rho}_{xA0}$$

For the part (DC):

$$\bar{\rho}_{yDC} = \frac{1}{R}\int_{-R}^0 \rho_y dy = \frac{1}{R}\int_{-R}^0 \frac{1}{dy/dx}\rho_x dy =$$

$$= \frac{\rho_x}{R}\int_{-R}^0 dx = \frac{\rho_x}{R}\int_0^R dx = \frac{\rho_x}{R}R = \rho_x = \bar{\rho}_{x0A}$$

At the end:

$$\bar{\rho}_y = \bar{\rho}_{yAB} = \bar{\rho}_{yBC} = \bar{\rho}_{yAD} = \bar{\rho}_{yDC} = \bar{\rho}_x = \rho_x$$

3. Quality of arrays {x} and {y}.

Let consider the function: y = sin x = sin φ, defined in the interval x = φ € [0, π/2]. In this interval it is continuous and monotonous. The graph of the function is presented on Figure 7.

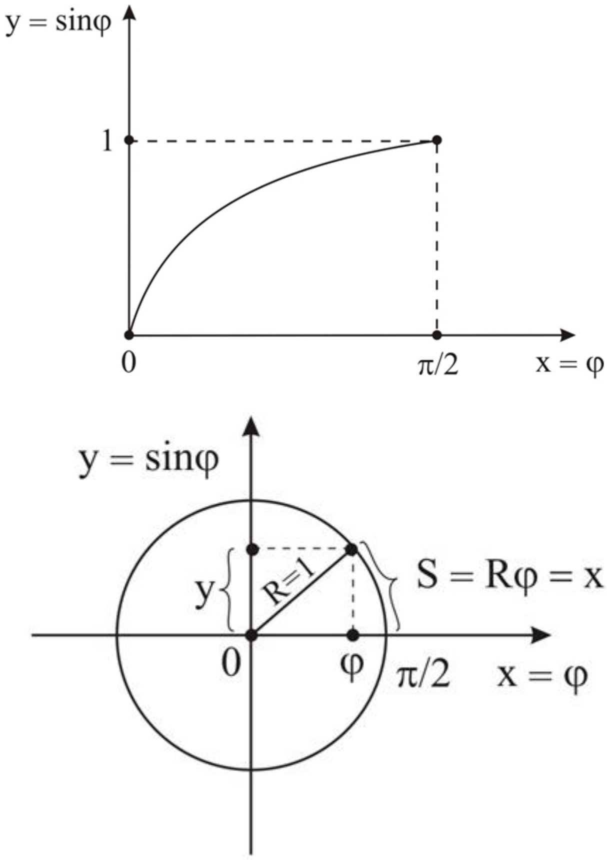


Figure 7. Equal quality of arrays {x} and {y}

In accordance with conditions the arrays {x} and {y} must have one and the same quality. On the vertical axis (ordinate) we have a length as part of the radius of the trigonometrically circle with R = 1. On the x axis (abscissa) we have the value of the angle φ. However, it can be measured in [deg], [rad], [grad], etc. In order the array {x} to have the same quality as {y}, its values must reciprocate to length. This is possible only if the angle φ is measured in [rad]. Then the perimeter of circle S, corresponding to φ is: S = Rφ = x and the array {x} has the same quality (length) as the array {y}. (Measurement of φ in other units will violate this condition). Then: y = R sinφ and x = S = R φ, as R = 1.

Accordingly, we shall have:

$$\rho_x = \bar{\rho}_x = \text{const}, \quad \rho_y = \text{var}$$

$$\rho_y = \frac{1}{|dy/d\varphi|}\rho_x = \frac{1}{|\cos\varphi|}\rho_x \geq \rho_x$$

$$\bar{\rho}_y = \frac{1}{1}\int_0^1 \rho_y dy = \int_0^1 \frac{1}{|dy/dx|}\rho_x dy =$$

$$= \rho_x \int_0^1 dx = \rho_x \int_0^{\pi/2} dx = \rho_x \frac{\pi}{2} > \rho_x = \bar{\rho}_x$$

In such ways we can expand the concept of density of function and argument.

C. Density of function of two variables.

Let consider an arbitrary, continuous, differentiable and monotonous function of two variables: z = f (x, y), defined in the intervals $[x_1, x_2]$ and $[y_1, y_2]$ (Figure 8) [1]. The graph of the function is a monotonous surface in space (x, y, z) with an area S_z, and the array of values of the arguments is plain lying in the plane (0xy) with an area S_{xy}.

On each point of the plane of the argument: (N_{xi}, N_{yi}) corresponds mutually and uniquely, exactly specified value of the function N_{zi}. Similarly with the function of one variable can be defined concepts average surface absolute density of the function and its arguments:

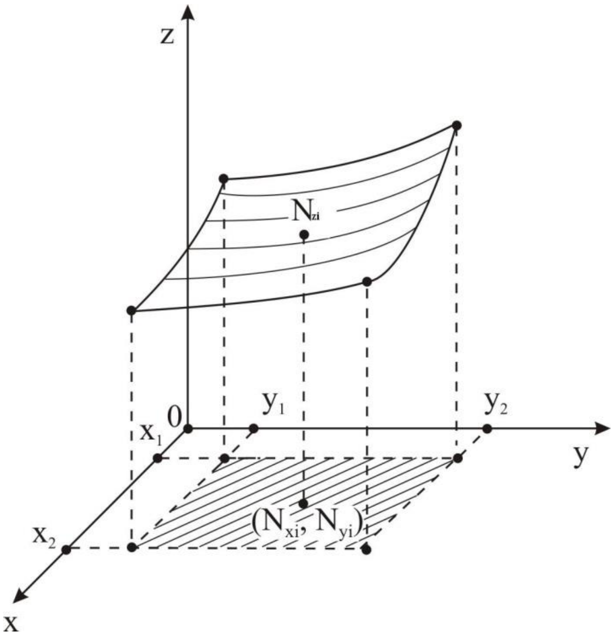


Figure 8. Density of function of two variables

$$\bar{\rho}_z = \frac{\{N_{zi}\}}{S_z}$$

$$\bar{\rho}_{xy} = \frac{\{N_{xi}, N_{yi}\}}{S_{xy}}$$

as:

$$\frac{\bar{\rho}_z}{\bar{\rho}_{xy}} = \frac{S_{xy}}{S_z} \leq 1$$

(of all surfaces the smallest area has the plane), differential absolute density:

$$d\rho_z = \frac{\{dN_{zi}\}}{dS_z}, \quad d\rho_{xy} = \frac{\{d(N_{xi}, N_{yi})\}}{dS_{xy}}$$

$$\frac{d\rho_z}{d\rho_{xy}} = \frac{dS_{xy}}{dS_z} \leq 1$$

and relative density of one function to another function:

$$\bar{\rho}_{1z} = \frac{\{N'_{zi}\}}{S'_z}, \quad \bar{\rho}_{2z} = \frac{\{N''_{zi}\}}{S''_z}$$

$$\frac{\bar{\rho}_{2z}}{\bar{\rho}_{1z}} = \frac{S'_z}{S''_z} \gtreqless 1$$

$$\frac{d\rho_{2z}}{d\rho_{1z}} = \frac{dS'_z}{dS''_z} \gtreqless 1$$

Everywhere areas S_{xy} and S_z will be calculated with surface integrals.

If the function: z = f (x, y) is not monotonous, the concept of average density is effective for those subintervals in which it is monotonous, but the differential densities preserve. Here unlimited intervals can be considered as well. The quality of the arrays {x, y} and {z} must be with the same dimensions of the surface.

D. Density of Function of Three Variables.

Everything said until now here is easily generalized for arbitrary, continuous, differentiable and monotonous function of three variables: f (x, y, z), which is defined in the spatial area of its arguments: $[x_1, x_2]$, $[y_1, y_2]$, $[z_1, z_2]$, representing parallelepiped with volume V_{xyz} [1]. The graph of such a function is a spatial body with volume $V_f \geq V_{xyz}$ in a 4-dimensional space. Respectively volume densities of the function and its arguments can be defined in the following type:

$$\bar{\rho}_f = \frac{\{N_{fi}\}}{V_f} \quad , \quad \bar{\rho}_{xyz} = \frac{\{N_{xi} N_{yi} N_{zi}\}}{V_{xyz}}$$

$$\frac{\bar{\rho}_f}{\bar{\rho}_{xyz}} = \frac{V_{xyz}}{V_f} \leq 1$$

$$d\rho_f = \frac{\{dN_{fi}\}}{dV_f} \quad , \quad d\rho_{xyz} = \frac{\{d(N_{xi} N_{yi} N_{zi})\}}{dV_{xyz}}$$

$$\frac{d\rho_f}{d\rho_{xyz}} = \frac{dV_{xyz}}{dV_f} \leq 1$$

and relative density of one function to another function:

$$\bar{\rho}_{1f} = \frac{\{N'_{fi}\}}{V'_f} \quad , \quad \bar{\rho}_{2f} = \frac{\{N''_{fi}\}}{V''_f}$$

$$\frac{\bar{\rho}_{2f}}{\bar{\rho}_{1f}} = \frac{V'_f}{V''_f} \gtreqless 1$$

$$\frac{d\rho_{2f}}{d\rho_{1f}} = \frac{dV'_f}{dV''_f} \gtreqless 1$$

Everywhere volumes V_{xyz} and V_f shall be calculated with volume integrals.

And here, as well, if the function: f (x, y, z) is not monotonous, the concept of average density is meaningful only for those intervals where it is of monotonous sense, and the differential densities preserve. Intervals can be illimitable. The quality of the arrays {f} and {x, w, z} must be one and the same – with dimension of volume.

E. Theorems of the Projections.

We shall prove two important theorems for projections of vector quantities and tensor quantities:

A. Theorem for the projections of vector quantity.

The projections of each vector **AB** on the coordinate axes have the average densities greater than or equal to the density of the vector itself:

$$\bar{\rho}_{AB} \leq \bar{\rho}_{AB_x}, \quad \bar{\rho}_{AB} \leq \bar{\rho}_{AB_y}, \quad \bar{\rho}_{AB} \leq \bar{\rho}_{AB_z}$$

The **proving** is simple, considering that each vector **AB** represents directed segment with a length **|AB|**, which has a length greater than or equal to the length of its projections on the coordinate axes.

$$|AB| \geq |AB_x|, |AB_y|, |AB_z|$$

B. Theorem for the projections of tensor quantity of second rank.

The projections of each tensor of second rank on the coordinate axes have average densities greater than or equal to the density of the tensor itself.

Proving: Each tensor of second rank can be divided into symmetrical and anti-symmetrical part [2]. The symmetrical part is presented in three-dimensional space with tri-axial ellipsoid whose projections on the coordinate planes 0xy, 0yz, 0xz have higher average densities from the projections of the ellipsoid on the plains of its own coordinate system: 0ab, 0bc, 0ac, where a, b, c are its semi-axles. Anti-symmetrical part of the tensor is represented by a vector, for which we have already proved that its projections on the coordinate axes have density greater or equal to the average density.

III. RESULT AND DISCUSSION

The introduced density of function and density of argument are new qualities in the mathematical analysis. They are holding a true when the function has the following conditions:

1. The function is continuous and differentiable in an arbitrary interval.
2. The function to be monotonous in this interval.
3. The arrays of values of the argument and function to have equal quality.

These qualities have many applications in the mathematics and physics.

The authors dedicate this work to Tzvetana Nedeva a true friend and colleague- mathematics.

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