International Journal of Scientific Research in Science and Technology
Available online at : www.ijsrst.com
OPEN ACCESS
Print ISSN: 2395-6011 | Online ISSN: 2395-602X
doi : https://doi.org/10.32628/IJSRST

An Enhanced Darbo-Type Fixed Point Theorems and Application to Integral Equations

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ARTICLEINFO

Article History:

Accepted : 01 Nov 2024
Published : 15 Nov 2024

Publication Issue :
Volume 11, Issue 6
November-December-2024

Page Number :
120-130

ABSTRACT

This manuscript introduces a generalized operator and presents new Darbo-type fixed point theorems pivotal in the existence theory of integral and differential equations. The significance of these theorems lies in their ability to provide conditions under which solutions to complex mathematical problems can be guaranteed. We establish our results by employing the measure of noncompactness within the context of Banach spaces, a framework that allows for a comprehensive analysis of functional equations. Our findings extend existing Darbo-type fixed point theorems and offer a deeper understanding of the underlying mathematical structures. By generalizing these results, we contribute to the broader field of fixed-point theory, enhancing its applicability to various mathematical disciplines. The implications of our work are substantial, as they facilitate the development of new methods for solving integral and differential equations that arise in both theoretical and applied contexts.

Keywords : Generalized Operator, Existence Theory, Integral Equations, Differential Equations, Measure of Noncompactness, Banach Spaces, Functional Equations, Mathematical Structures, Solution Guarantees, Theoretical Mathematics

I. INTRODUCTION

Integral equations are fundamental for formulating and analyzing various mathematical models across disciplines. Specifically, Volterra integral equations have been instrumental in capturing the dynamics of complex systems, such as the motion of particles in a fluid medium, population dynamics, and other areas where delays and memory effects are significant. These equations are distinguished by their ability to incorporate delays proportional to time or other variables. They are particularly relevant for models where future states depend on past values non-instantaneously. Solving these equations presents unique challenges, prompting the development and application of various solution methodologies. Researchers have employed numerous analytical and numerical methods to address the complexity of these integral equations. Techniques like the Walsh, variational iteration, and homotopy perturbation methods have been extensively studied for their effectiveness in solving systems of integral equations. Each approach has specific advantages depending on the structure and properties of the equation being analyzed. These methods, along with others referenced in recent literature, provide a range of tools that researchers can select based on the problem's requirements and the desired precision of the solution [2,7,8].

However, one approach that has garnered significant attention in recent years is Darbo's fixed-point theorem. Based on fixed-point theory, this method has shown exceptional promise in proving the existence and uniqueness of solutions for integral equations, particularly in cases involving nonlinearities and proportional delays. Darbo's fixed-point theorem is versatile because it leverages the concept of the Measure of Non-Compactness (MNC), effectively handling equations with complex structures that may be challenging to solve directly. As a result, the Darbo fixed point approach has become a focal point of interest for researchers working on complex differential and integral equations, leading to various generalizations and extensions of the theorem. These extensions have broadened the applicability of Darbo's approach, enhancing its relevance for solving integral equations that arise in fields as diverse as mathematical biology, physics, and engineering [2-10]. In this study, we contribute to this growing body of research by defining a new class of condensing operators and establishing corresponding fixed-point results within the framework of Darbo's theorem. Condensing operators are essential tools in fixed-point theory, particularly in analyzing equations involving compact properties or exhibiting complex delay structures. By developing new fixed-point results for these operators, we aim to expand the applicability of Darbo's theorem to a broader class of integral equations, including those with nonlinear. These theoretical advancements are then employed to establish the existence of solutions for a specific type of nonlinear Volterra integral equation, thus demonstrating the utility of our approach in tackling challenging real-world problems.

Definition 1(Measure of Noncompactness) [7]

A function $\sigma\colon \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is called M.N.C provided it fulfils the following axioms:

i) (Regularity) $\sigma(W) = 0$ if and only if W is relatively compact.

ii) The family $\ker \sigma = \{W \in \mathfrak{M}_E : \sigma(W) = 0\}$ is a non-empty and $\ker \sigma \subseteq \mathfrak{N}_E$.

iii) (Monotonicity) $W \subset W' \Longrightarrow \sigma(W) \le \sigma(W')$.

iv) (Invariant under closure) $\sigma(W) = \sigma(\bar{W})$.

v) (Invariant under convex hull) $\sigma(W) = \sigma(coW)$.

vi) $\sigma(\alpha W + (1-\alpha)W') \le \alpha\sigma(W) + (1-\alpha)\sigma(W')$, for all $\alpha \in [0,1]$.

vii) (Generalized Cantor's intersection theorem) If $W_n \in \mathfrak{M}_E$ for $n = 1,2,\cdots$ is decreasing sequence of closed subsets of E and $\lim_{n\to\infty}\sigma(W_n) = 0$ then $W_\infty = \bigcap_{n=1}^{\infty} W_n$ is non-empty.

The family is defined in axiom (i) is called the kernel of the M.N.C and denoted by ker. In fact, by axiom (vi), we have $\sigma(\Omega_\infty) \le \sigma(\Omega_n)$ for any n, thus $\sigma(\Omega_\infty) = 0$. This yields that $\Omega_\infty \in \ker\ \sigma$.

Theorem 1.1. (Schauder's fixed point theorem [12]) Let Ω be a member of the class N.B.C.C of a Banach space E, then every continuous and compact mapping on Ω has at least one fixed point in Ω.

Darbo's fixed-point theorem concerning an M.N.C σ can be stated as below.

Theorem 1.2. (Darbo's fixed point theorem [1]) Let Ω be a member of the class N.B.C.C of a Banach space E

and T be the continuous self-mapping defined on be a continuous mapping such that for any nonempty subset W of Ω,

$$\sigma(T(W)) \leq \lambda\sigma(W) \qquad (1)$$

for some $\lambda \in [0,1)$ and every non-empty subset W of Ω. Then T has at least one fixed point in Ω.

Definition 2. Let $C_{\Im}$ denotes the class of all the functions $\Im : \mathbb{R}_+ \to \mathbb{R}_+$ and C_H be the class of functions $H(\bullet;.) : C_{\Im} \to C_{\Im}$\$, which validates the following assumptions:

$$[H_i)] H(\Im;u) > 0 \; for u > 0 and H(\Im;0) = 0.$$
$$[H_{ii})] H(\Im;u) \leq H(\Im;v) \; for u \leq v.$$
$$[H_{iii})] \lim_{n\to\infty} H(\Im;u_n) = H\left(\Im;\lim_{n\to\infty} u_n\right).$$
$$[H_{iv})] H(\Im;\max\{u,v\}) = \max\{H(\Im;u), H(\Im;v)\} \; for\ some\ \Im \in C_{\Im}.$$

Definition 3. Let C_F be the class of functions $F : [0,\infty) \times : [0,\infty) \to \mathbb{R}$ that validates the assumptions:

$[F_i)] F(u,v) \leq u$

$[F_{ii})] F(u,v) = u \; implies\ either\ u = 0\ or\ v = 0;\ for\ all\ u,v \in [0,\infty).$

Example 1. The following functions are the members of the class C_F, $for\ all\ u,v \in [0,\infty)$:

$[(1)] F(u,v) = \lambda u, 0 < \lambda < 1$

$[(2)] F(u,v) = u - v$

$[(3)] F(u,v) = \Theta(u), where\ \Theta : [0,\infty) \to [0,\infty)\ is\ upper\ semicontinuous\ function\ such\ that\ \Theta(0) = 0, and\ \Theta(u) < u\ for\ u > 0.$

$[(4)] F(u,v) = u - \theta(u)\ where\ \theta : [0,\infty) \to [0,\infty)\ is\ continuous\ function\ such\ that\ \theta(u) = 0\ then\ u = 0.$

Definition 4. Let C_Q be the class of all the continuous functions $Q : [0,\infty) \times : [0,\infty) \to \mathbb{R}$ \$, that validates the following assumptions;

$[Q_i)] Q\ is\ non-decreasing.$

$[Q_{ii})] Q(0) = 0.$

II. MAIN RESULT

Theorem 2.1. Let K be an arbitrary member of the class $N.B.C.C$ of a Banach space B and $T : K \to K$ is a continuous function. If S be any non-empty subset of K such that

$$Q\left[R\left(\int_0^{\aleph(TS)} \zeta(\tau)d\tau\right)\right] \leq F\left[Q\left[R\left(\int_0^{\aleph(S)} \zeta(\tau)d\tau\right)\right], G\left[R\left(\int_0^{\aleph(TS)} \zeta(\tau)d\tau\right)\right]\right], \qquad (2)$$

Where, $\aleph$ is $M.N.C$, $F \in C_F$, $Q \in C_Q$, $G,Q : [0,\infty) \to [0,\infty)$ with $\lim_{u\to 0} G(u) = 0 \Rightarrow u = 0$, and $\zeta : (0,\infty) \to (0,\infty)$ $R : [0,\infty) \to [0,\infty)$ are continuous functions with $\lim_{u\to 0} R(u) = 0 \Rightarrow u = 0$. Then T admits at least one fixed point in K.

Proof. Consider the sequence of sets $\langle D_n \rangle$ defined by

$$S_{n+1} = \overline{conv} T(S_n) \text{ by the initial approximation } S_0 = K.$$

Since, $T(S_0) \subseteq S_0$ thus

$S_1 = \overline{conv}T(S_0) \subseteq S_0$, continuing in this fashion we get

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n,$$

and

$$S_{n+1} = \overline{conv}T(S_n) \subseteq \overline{conv}T(S_{n-1}) = S_n.$$

Hence by mathematical induction, the sequence $S_{n+1} = \overline{conv}T(S_n)$ of subsets of K is non-increasing, i.e., $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$.

If there exists a non-negative integer n such that $\aleph(S_n) = 0$ then S_n is precompact set and $T(S_n) \subseteq S_n$. Thus Theorem 1, implies that T has at least one fixed point in K.\\

Now, on the another end we may assume $\aleph(S_n) > 0$ for all $n \in \mathbb{N}$.

Claim that $\left\langle Q\left[H\left(\Im; R\left(\int_0^{\aleph(S_n)} \zeta(\tau)d\tau\right)\right)\right]\right\rangle$ is a non-increasing sequence. for all n >1. The monotonic decreasing property of the sequence $\langle S_n \rangle$ and axiom (iii) of Definition 1 implies that $\langle \Theta(\aleph(S_n)) \rangle$ is a monotonic decreasing sequence of real numbers. Since the function Θ is bounded below, there exist $\ell \in \mathbb{R}_+$ such that $\lim_{n\to\infty} \Theta(\aleph(S_n)) = \ell.$

If possible, assume that

$$Q\left[H\left(\Im; R\left(\int_0^{\aleph(S_{n0})} \zeta(\tau)d\tau\right)\right)\right] \le Q\left[H\left(\Im; R\left(\int_0^{\aleph(S_{n0+1})} \zeta(\tau)d\tau\right)\right)\right], \tag{3}$$

for some $n_0 \in \mathbb{N}$. Then by the virtue of inequality (3), we have

$$Q\left[H\left(\Im; R\left(\int_0^{\aleph(TS_{n0})} \zeta(\tau)d\tau\right)\right)\right] \le F\left[\begin{array}{l} Q\left[H\left(\Im; R\left(\int_0^{\aleph(S_{n0})} \zeta(\tau)d\tau\right)\right)\right], \\ G\left[H\left(\Im; R\left(\int_0^{\aleph(TS_{n0})} \zeta(\tau)d\tau\right)\right)\right] \end{array}\right]$$

$$Q\left[H\left(\Im; R\left(\int_0^{\aleph(\overline{conv}TS_{n0})} \zeta(\tau)d\tau\right)\right)\right] \le F\left[\begin{array}{l} Q\left[H\left(\Im; R\left(\int_0^{\aleph(S_{n0})} \zeta(\tau)d\tau\right)\right)\right] \\ G\left[H\left(\Im; R\left(\int_0^{\aleph(\overline{conv}TS_{n0})} \zeta(\tau)d\tau\right)\right)\right] \end{array}\right]$$

$$Q\left[H\left(\Im; R\left(\int_0^{\aleph(S_{n0+1})} \zeta(\tau)d\tau\right)\right)\right] \le F\left[\begin{array}{l} Q\left[H\left(\Im; R\left(\int_0^{\aleph(S_{n0})} \zeta(\tau)d\tau\right)\right)\right], \\ G\left[H\left(\Im; R\left(\int_0^{\aleph(S_{n0+1})} \zeta(\tau)d\tau\right)\right)\right] \end{array}\right] \tag{4}$$

using the assumption F_{ii}, we get

$$Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_{n_0+1})}\zeta(\tau)d\tau\right)\right)\right]\leq Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_{n_0})}\zeta(\tau)d\tau\right)\right)\right], \tag{5}$$

which is a contradiction to (2). This shows that

$$Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_{n+1})}\zeta(\tau)d\tau\right)\right)\right]\leq Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_{n})}\zeta(\tau)d\tau\right)\right)\right],$$

for all $n\in\mathbb{N}$. Thus $\left\langle Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_{n})}\zeta(\tau)d\tau\right)\right)\right]\right\rangle$, is the monotonic decreasing sequence of non-negative real numbers. Using the monotonic property of Q, we get

$$H\left(\Im;R\left(\int_0^{\aleph(S_{n+1})}\zeta(\tau)d\tau\right)\right)\leq H\left(\Im;R\left(\int_0^{\aleph(S_{n})}\zeta(\tau)d\tau\right)\right). \tag{6}$$

This shows that $\left\langle H\left(\Im;R\left(\int_0^{\aleph(S_{n})}\zeta(\tau)d\tau\right)\right)\right\rangle$, is a monotonic decreasing sequence of positive real numbers.

Hence there exist a non-negative real number ℓ such that $\lim_{n\to\infty} H\left(\Im;R\left(\int_0^{\aleph(S_{n})}\zeta(\tau)d\tau\right)\right)=\ell.$

Now, we prove that $\ell=0$ Assume the contradiction that $\ell>0$. By the virtue of inequality (2), we get

$$\begin{aligned}
Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_{n+1})}\zeta(\tau)d\tau\right)\right)\right] &\leq Q\left[H\left(\Im;R\left(\int_0^{\aleph(\overline{conv}TS_n)}\zeta(\tau)d\tau\right)\right)\right]\\
&\leq Q\left[H\left(\Im;R\left(\int_0^{\aleph(TS_n)}\zeta(\tau)d\tau\right)\right)\right]\\
&\leq F\left[\begin{matrix} Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_n)}\zeta(\tau)d\tau\right)\right)\right], \\ G\left[H\left(\Im;R\left(\int_0^{\aleph(TS_n)}\zeta(\tau)d\tau\right)\right)\right]\end{matrix}\right]
\end{aligned}$$

The above inequality along with assumption (F_{ii}), takes following form

$$Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_{n+1})}\zeta(\tau)d\tau\right)\right)\right]\le F\left[\begin{array}{l}Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_n)}\zeta(\tau)d\tau\right)\right)\right],\\ G\left[H\left(\Im;R\left(\int_0^{\aleph(TS_n)}\zeta(\tau)d\tau\right)\right)\right]\end{array}\right]$$

$$\le Q\left[H\left(\Im;R\left(\int_0^{\aleph(S_n)}\zeta(\tau)d\tau\right)\right)\right].$$

Applying the limit as $n\to\infty$, we get $\ell\le F\left[\ell,G(\ell)\right]\le\ell$, which implies $F\left[\ell,G(\ell)\right]=\ell$, from assumption (F_i) and behaviour of G at zero confirm that $\ell=0$. Consequently, the preoperty of R implies that

$\lim\limits_{n\to\infty}\int_0^{\aleph(S_n)}\zeta(\tau)d\tau=0.$ Note that for any $\delta>0,\int_0^{\delta}\zeta(\tau)d\tau>0,$ this ensures that $\aleph(S_n)\to 0\; when\; n\to\infty.$

Since $S_{n+1}\subseteq S_n$ *for all* $n\in$ □ i.e., $\langle S_n\rangle$ is a decreasing sequence of closed and bounded nested sets. By the axiom vii) of definition, the countable intersection $S_\infty=\bigcap_{n=1}^{\infty}S_n$, is nonempty, closed,

convex and compact. We assure the existence of a fixed point in the view of Schauder fixed point theorem for $T:S_\infty(\subset K)\to S_\infty$.

Following are some immediate consequences of Theorem 2.1. If we take $H(\bullet;.):C_\Im\to C_\Im$ as $H(\Im,R(u))=R(u)$ for a continuous function $R:[0,\infty)\to[0,\infty)$, then we get the following consequence. □

Theorem 2.2. Let K be an arbitrary member of the class N.B.C.C of a Banach space B and $T:K\to K$ is a continuous function. If S be any non-empty subset of K such that

$$Q\left[R\left(\int_0^{\aleph(TS)}\zeta(\tau)d\tau\right)\right]\le F\left[Q\left[R\left(\int_0^{\aleph(S)}\zeta(\tau)d\tau\right)\right],G\left[R\left(\int_0^{\aleph(TS)}\zeta(\tau)d\tau\right)\right]\right],\qquad(8)$$

where $\aleph$ is M.N.C, $F\in C_F$, $Q\in C_Q$, $G,:[0,\infty)\to[0,\infty)$ with $\lim\limits_{u\to 0}G(u)=0\Rightarrow u=0$, $\zeta:(0,\infty)\to(0,\infty)$ and $R:[0,\infty)\to[0,\infty)$ are continuous functions with $\lim\limits_{u\to 0}R(u)=0\Rightarrow u=0.$ Then T admits at least one fixed point in K.

If we take $F(u,v)=u-v$ *and* $G,R:[0,\infty)\to[0,\infty)$ as $G(u)=R(u)=u$ in Theorem 2.1, we get the following consequence. □

Theorem 2.3. Let K be an arbitrary member of the class N.B.C.C of a Banach space B and $T:K\to K$ is a continuous function. If S be any non-empty subset of K such that

$$H\left(\Im;\int_0^{\aleph(T(S))}\zeta(\tau)d\tau\right)\le Q\left[H\left(\Im;\int_0^{\aleph(S)}\zeta(\tau)d\tau\right)\right]-Q\left[H\left(\Im;\int_0^{\aleph(T(S))}\zeta(\tau)d\tau\right)\right],\qquad(9)$$

where $\aleph$ is M.N.C, $H(\bullet;.)\in C_H$, $Q\in C_Q$ and $\zeta:(0,\infty)\to(0,\infty)$ is a continuous function. Then T admits at least one fixed point in K.

If we take $F\in C_F$ as $F(u,v)=u-v$, and $H(\bullet;.)\in C_H$ as $H(\Im;u)=u$ and $R:[0,\infty)\to[0,\infty)$ as $R(u)=u$. Then we get the following consequence. □

Theorem 2.4. Let K be an arbitrary member of the class N.B.C.C of a Banach space B and $T:K\to K$ is a continuous function. If S be any non-empty subset of K such that

$$Q(\aleph(TS)),, Q(\aleph(S))-G(\aleph(TS)), \tag{10}$$

where $\aleph$ is M.N.C, $G,Q,:[0,\infty)\to[0,\infty)$ are continuous function with $\lim_{u\to 0}Q(u)=0\Rightarrow u=0$ and $\lim_{v\to 0}G(v)=0\Rightarrow v=0$. Then T admits at least one fixed point in K. □

III. COROLLARY

Following are the corollaries can found in the literatures.

Corollary 3.1. Let K be an arbitrary member of the class N.B.C.C of a Banach space B and $T:K\to K$ is a continuous map. If for any non empty subset S of K

$$H\left(\Im;\int_0^{\aleph(TS)}\zeta(\tau)d\tau+\phi\left(\int_0^{\aleph(TS)}\zeta(\tau)d\tau\right)\right)\le Q\left[H\left(\Im;\int_0^{\aleph(S)}\zeta(\tau)d\tau+\phi\left(\int_0^{\aleph(S)}\zeta(\tau)d\tau\right)\right)\right] -Q\left[H\left(\Im;\int_0^{\aleph(TS)}\zeta(\tau)d\tau+\phi\left(\int_0^{\aleph(TS)}\zeta(\tau)d\tau\right)\right)\right], \tag{11}$$

where $\aleph$ is M.N.C, $H(\bullet;.)\in C_H$, $Q\in C_Q$, $\zeta,\phi:(0,\infty)\to(0,\infty)$ are continuous functions. Then T admits at least one fixed point in K.

Proof. In Theorem 1, if we take $F:[0,\infty)\times[0,\infty)\to[0,\infty)$ as

$$F(u,v)=u-v, \Phi:[0,\infty)\to[0,\infty)\, as\, \Phi(u)=u+\phi(u)$$

we get the above corollary.

Corollary 3.2. Let K be an arbitrary member of the class N.B.C.C of a Banach space B and $T:K\to K$ is a continuous map. If for any non empty subset S of K

$$H(\Im;\aleph(TS)+\phi(\aleph(TS)))\le\psi(H(\Im;\aleph(S)+\phi(\aleph(S)))), \tag{12}$$

where $\aleph$ is M.N.C, $H(\bullet;.)\in C_H$, $Q\in C_Q$, and $\psi:[0,\infty)\to[0,\infty)$, and $\phi:(0,\infty)\to(0,\infty)$ is a monotonic increasing function and $\lim_{n\to\infty}\psi^n(u)=0$. Then T admits at least one fixed point in K.

Proof. In Theorem 2.1 if we take $F:[0,\infty)\times[0,\infty)\to[0,\infty)$ as $F(u,v)=\psi(u)$, and $Q,\zeta:[0,\infty)\to[0,\infty)$ $Q(u)=u,\zeta(u)=1$ and $R:[0,\infty)\to[0,\infty)$ as $R(u)=u+\phi(u)$, then we get above the corollary. □

Corollary 3. Let K be an arbitrary member of the class N.B.C.C of a Banach space B and $T:K\to K$ is a continuous map. If for any non empty subset S of K

$$H(\Im;\aleph(TS)+\phi(\aleph(TS)))\le\lambda(H(\Im;\aleph(S)+\phi(\aleph(S)))), \tag{13}$$

where $\aleph$ is M.N.C, $H(\bullet;.)\in C_H$, $Q\in C_Q$, and $\phi:(0,\infty)\to(0,\infty)$ is a monotonic increasing function. Then T admits at least one fixed point in K.

Proof. In Theorem 2.1, if we take $\mathrm{F}:[0,\infty)\times[0,\infty)\to[0,\infty)$ as $\mathrm{F}(u,v)=\lambda u$, $\mathrm{Q},\zeta:[0,\infty)\to[0,\infty)$ as $\mathrm{Q}(u)=u,\zeta(u)=1$ and $\mathrm{R}:[0,\infty)\to[0,\infty)$ as $\mathrm{R}(u)=u+\phi(u)$, then we get above the corollary. □

Corollary 4. Let K be an arbitrary member of the class N.B.C.C of a Banach space B and $\mathrm{T}:\mathrm{K}\to\mathrm{K}$ is a continuous map. If for any non-empty subset S of K

$$\mathrm{Q}\left(\int_0^{\aleph(\mathrm{TS})}\zeta(\tau)d\tau\right)\le\mathrm{F}\left[\mathrm{Q}\left(\int_0^{\aleph(\mathrm{S})}\zeta(\tau)d\tau\right),\mathrm{G}\left(\int_0^{\aleph(\mathrm{TS})}\zeta(\tau)d\tau\right)\right],\tag{14}$$

where $\aleph$ is M.N.C, $\mathrm{F}\in\mathcal{C}_{\mathrm{F}}$, $\mathrm{Q}\in\mathcal{C}_{\mathrm{Q}}$, $\mathrm{G},:[0,\infty)\to[0,\infty)$ with $\lim_{u\to 0}\mathrm{G}(u)=0\Rightarrow u=0$, $\zeta:(0,\infty)\to(0,\infty)$ is a continuous function. Then T admits at least one fixed point in K.

Proof. In Theorem 2.2, if we take $\mathrm{R}:[0,\infty)\to[0,\infty)$ as $\mathrm{R}(u)=u$, then we get the above corollary.

IV. Application

Over the last few decades, many researchers have shown that the concept of M.N.C brings into play a sparkling role in studying the existence and uniqueness of the solution of integral equations. In this section, we will use the M.N.C in the space $C([0,a])$ contains all continuous functions $x:[0,a]\to\mathbb{R}$ h having the norm,

$$\|x\|=\max\{|x(r)|:r\in[0,a]\};x\in C([0,a]),\tag{15}$$

Let $\Lambda\neq\phi$ be any subset of $C([0,a];\mathbb{R})$. Now for $\epsilon>0$ we define the modulus of continuity $\omega(x,\epsilon)$ of x on $[0,a]$ as;

$\omega(x,\varepsilon)=\max\{|x(r)-x(s)|;r,s\in[0,a],|r-s|\le\varepsilon\}$, and further, we define the term $\omega(W,\varepsilon)$ as follows; $\omega(\Lambda,\varepsilon)=\sup\{\omega(x,\varepsilon);x\in\Lambda\}$. Note that the modulus of continuity $\omega(\Lambda,\varepsilon)$, is non-negative and increasing; hence we ensure that there exists a finite limit of $\lim_{\varepsilon\to 0}\omega(\Lambda,\varepsilon)$, and finally, we obtain the expression for the term $\mathrm{M}(\Lambda)$ in the form of limits as;

$$\mathrm{M}(\Lambda)=\lim_{\varepsilon\to 0}\omega(\Lambda,\varepsilon).\tag{16}$$

In [7, 9] it is proved that the term M mentioned in equation (15) is M.N.C in the Banach space

Now, we use the results from section 2 to prove a theorem which ensures the solutions of the nonlinear Volterra integral equation.

$$Q(t)=\mathsf{F}(t,Q(t))+\int_0^t\mathrm{A}(t,\tau,Q(\tau))d\tau\quad t\in I=[0,1]\tag{17}$$

To prove the existing result, we need to consider the following assumptions

1) $\mathsf{F}:I\times\mathbb{R}\to\mathbb{R}$, is continuous function satisfies the following inequality;

   $|\mathsf{F}(t,Q_1(t))-\mathsf{F}(t,Q_2(t))|\le\Delta|Q_1(t)-Q_2(t)|$ where, Δ is a real number.

2) $\mathrm{A}:I\times[0,1]\times\mathbb{R}\to\mathbb{R}$, is a continuous function and a there exists non-decreasing $\eta:\mathbb{R}\to\mathbb{R}_+$, such that $|\mathrm{A}(t,\tau,Q(\tau))|\le\eta(\|Q\|)$.

3) There is at least one real number $r_0 > 0,$ which sacrifices the inequality

$$\eta(r_0)+\Delta(r_0)+\mathrm{M} \leq r_0,$$

where $\mathrm{M} \geq 0$\$, is constant, which satisfies the relation $\left|\mathrm{F}(t,0,)\right| \leq \mathrm{M}$ for all $t \in I$.

Theorem 4.1. If assumptions 1-3 are satisfied by the Volterra integral equation (17), then it has at least one solution in the space $C([0,a])$.

Proof. For $Q \in C(I)$ define an mapping Q on the Banach space C(I) in the following manner;

$$\mathrm{P}(Q(t)) = \mathrm{F}(t,Q(t)) + \int_0^t \mathrm{A}(t,\tau,Q(\tau))\,d\tau \tag{18}$$

In this case, it is simple to demonstrate that P is a self-mapping on the space $C(I)$. Without sacrificing generality, we accept s < t, and find the following equation to do this fix, $\epsilon > 0,$ and select random integers $t,s \in I$ such that $|t-s| < \epsilon$.

$$\begin{aligned}\left|\mathrm{P}(Q(t))-\mathrm{P}(Q(s))\right| &\leq \left|\mathrm{F}(t,Q(t))+\int_0^t \mathrm{A}(t,\tau,Q(\tau))\,d\tau-\mathrm{F}(s,Q(s))-\int_0^s \mathrm{A}(s,\tau,Q(\tau))\,d\tau\right| \\ &\leq \left|\mathrm{F}(t,Q(t))-\mathrm{F}(s,Q(s))\right|+\left|\int_s^0 \mathrm{A}(s,\tau,Q(\tau))\,d\tau+\int_0^t \mathrm{A}(t,\tau,Q(\tau))\,d\tau\right| \\ &\leq \Delta\omega(Q,\varepsilon)+\eta(\|Q\|)|t-s|\end{aligned} \tag{19}$$

were, $\omega(Q,\varepsilon)=\sup\{|Q(t)-Q(s)| : t,s\in[0,1], |t-s| \leq \varepsilon\}$.

We make sure that PQ, is a continuous function on the interval I based on the estimate above and assumptions (i) & (ii). We now ensure that the mapping Q converts the space C(I) into itself by considering the previously stated equation and assumption (iii). Examine $B_{r_0} = \{u \in C([0,1]) \| u \| \leq r_0\}$, for $r_0 > 0$. be the closed ball with the origin in its centre.

We assert that Q is a continuous map into itself from B_{r_0} In fact, we ensure the following inequality for a random but fixed element $Q \in C(I)\, and\, t \in I$:

$$\begin{aligned}\left|P(Q(t))\right| &= \left|\mathrm{F}(t,Q(t))-\mathrm{F}(t,Q(t))+\mathrm{F}(t,Q(t))+\int_0^t \mathrm{A}(t,\tau,Q(\tau))\,d\tau\right| \\ &\leq \left|\mathrm{F}(t,Q(t))-\mathrm{F}(t,0)\right|+\left|\mathrm{F}(t,0)\right|+\left|\int_0^t \mathrm{A}(t,\tau,Q(\tau))\,d\tau\right| \\ &\leq \left|\mathrm{F}(t,Q_1(t))-\mathrm{F}(t,Q_2(t))\right|+\int_0^t\left|\mathrm{A}(t,\tau,Q_1(\tau))-\mathrm{A}(t,\tau,Q_2(\tau))\right|d\tau \\ &\leq \Delta\left|Q_1(t)-Q_2(t)\right|+\omega_{\mathrm{A}}(I,\varepsilon)t\end{aligned} \tag{20}$$

Equation (20) and assumption (iii) enabled us to conform Q mappings from the B_{r0} into itself.

To demonstrate that Q is continuous on B, fix $\epsilon > 0$ and select $\delta > 0$. At the same time, pick an arbitrary pair $Q_1, Q_2 \in B$ so that $\| Q_1 - Q_2 \| < \delta$. Taking this into account, for a random $t \in I$, we get;

$$\begin{aligned}\left|\mathrm{P}\left(Q_1(t)\right)-\mathrm{P}\left(Q_2(t)\right)\right| &\leq \left|\mathrm{F}\left(t,Q_1(t)\right)-\mathrm{F}\left(t,Q_2(t)\right)\right|+\int_0^t\left|\mathrm{A}\left(t,\tau,Q_1(\tau)\right)-\mathrm{A}\left(t,\tau,Q_2(\tau)\right)\right|d\tau \\ &\leq \Delta\left|Q_1(t)-Q_2(t)\right|+\omega_{\mathrm{A}}\left(I,\varepsilon\right)t\end{aligned} \tag{21}$$

Where ,$[\omega_{\mathrm{A}}\left(I,\varepsilon\right)=\sup\left\{\left|\mathrm{A}\left(t,u,v\right)-\mathrm{A}\left(s,u,v\right)\right|:t,s\in I,\left|t-s\right|\leq\varepsilon\right\}$.

The continuous quality of the mapping P on B_{r0} is confirmed by the equation (21).Since Q(t) > 0 for $t\in I$, let B be a collection of all the functions from the closed ball $Q\in B_{r_0}$. It is clear that B is not empty because $r_0 > 0$. Let $Q\in D$ and $D\neq\phi$, be non-empty subsets of B. Select the pair $t,s\in I$ such that $|t-s|\leq \epsilon$ for a specified real integer $\epsilon>0$. Assuming $s < t$, equation (19) yields the following expression without sacrificing generality:

$$\omega\left(PQ,\varepsilon\right)\leq\Delta\omega\left(PQ,\varepsilon\right)+\eta\left(\|Q\|\right)\left|t-s\right|. \tag{22}$$

By the virtue of assumptions (i) and (ii) the term $\omega_{\mathrm{K}}\left(I,\varepsilon\right)\rightarrow 0$ & $\omega_{\Re}\left(I,\varepsilon\right)\rightarrow 0$ as $\epsilon\rightarrow 0$, hence applying $\epsilon\rightarrow 0$ and from (16), equation (22) remain with the following inequality;

$$\Re\left(\mathrm{M}\left(P(D)\right)\right)\leq\Re\left(\mathrm{M}\left(D\right)\right)+\mathrm{G}\left(\mathrm{M}\left(P(D)\right)\right), \tag{23}$$

where M is M .N .C. The functions $\mathrm{G},\Re,:[0,\infty)\rightarrow[0,\infty)$ defined by $\Re\left(\hbar\right)=\Delta\hbar$ & $\mathrm{G}\left(\hbar\right)=\beta\hbar$ where β is non-negative real number such that $\Delta+\beta=1$.

With the preceding estimation and the theorem (2.4) of Section (2), we verify that the map P allows a fixed point in $B\subset C([0,1])$. This demonstrates that at least one solution to the integral equation (17) exists in Banach space $C\left(\left[0,\ 1\right]\right)$.

V. CONCLUSION

In this manuscript, we have introduced a novel generalized operator and established new Darbo-type fixed point theorems, which play a critical role in the existence theory of integral and differential equations. Utilizing the noncompactness measure within the Banach spaces framework provides a robust analytical foundation for exploring the solvability of complex functional integral equations. Our work extends the scope of existing Darbo-type fixed point theorems and deepens the understanding of the mathematical structures underlying these results. The generalizations presented here significantly enhance the versatility of fixed point theory, offering new perspectives and tools for addressing mathematical challenges in diverse fields. The theoretical advancements in this study have substantial implications for both pure and applied mathematics. They pave the way for developing innovative approaches to solving integral and differential equations, broadening fixed point theory's applications in science and engineering. These results will inspire further research, fostering advancements in analysing nonlinear equations and their practical applications.

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