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Optimal Retirement Wealth Allocation under Volatile Interest Rates: A GARCH-Based Analysis

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ABSTRACT

Managing post-retirement wealth effectively is crucial for ensuring financial security in uncertain market conditions. Traditional pension investment models assume constant interest rates, which fail to capture real-world financial volatility. This study develops an optimal investment strategy for post-retirement wealth management under stochastic interest rates, modeled using EGARCH and GJR-GARCH frameworks. By leveraging GARCH-type models, we estimate volatility dynamics and optimize asset allocation strategies. The Hamilton-Jacobi-Bellman (HJB) equation is applied within a stochastic control framework to derive the optimal investment policy. Sensitivity analysis is conducted to assess the impact of different risk aversion levels on portfolio allocation. The results demonstrate that accounting for stochastic interest rate volatility improves wealth sustainability in the post-retirement phase.

Keywords: GARCH models, HJB equation, Optimal asset allocation, Pension wealth management and Stochastic interest rates.

I. INTRODUCTION

Optimal retirement planning, especially within defined contribution pension schemes, requires tailored investment strategies that evolve through both wealth accumulation before retirement and wealth preservation afterward. The traditional models often assume constant interest rates, but this simplifying assumption limits their applicability in real-world conditions where interest rates are both volatile and asymmetric in response to economic shocks. Merton's foundational work on continuous-time portfolio optimization provided a valuable framework for dynamic investment strategies [Merton, 1971].

To address the limitations of constant interest rate, recent research has incorporated stochastic interest rate models to capture real-world rate fluctuations more accurately [Lioui, and Poncet, 2001; Korn and Kraft, 2001; Munk and Sørensen, 2004; Flor and Larsen, 2014; Lin and Riedel, 2021]. To adequately capture asymmetric responses and volatility clustering, the GARCH-type models, particularly EGARCH and GJR-GARCH, have been applied to model conditional volatility in interest rates. The EGARCH model captures asymmetric volatility without requiring positivity constraints [Nelson, 1991] and the GJR-GARCH model effectively captures leverage effects [Glosten et al., 1993].

This study focuses on comparing EGARCH and GJR-GARCH models in representing short-term interest rates volatility during post-retirement investment period. Manasi and Talawar (2025) have used GARCH type models in representing long-term interest rates for whole life pre-retirement investment period. Extending this framework, we apply stochastic optimal control techniques, which have been applied in pension fund modeling and annuity contracts [Devolder et al., 2003; Charupat and Milevsky, 2002; Osu and Ijioma, 2012; Mallappa and Talawar, 2019], to optimize post- retirement asset allocation strategies under stochastic interest rates.

II. METHODS AND MATERIAL

The liabilities after retirement are supposed to be paid in the form of an annuity whose level is guaranteed by the insurer. During the activity period, the contributions can be invested in a riskless or risky asset and the reserve obtained at retirement age is the amount accumulated without any special guarantee by the insurer. At the time of retirement this reserve is used to purchase a paid-up annuity. After retirement the insurer has to pay this guaranteed annuity.

The optimal control problem is defined as [see also Devolder et al., 2003],

The financial market is supposed to be described by two assets:

One of the assets is a savings account following the di erential equation

One of the assets is a savings account, that is, riskless asset with price dynamics S_1,following the differential equation,

$$dS_1(t) = r(t)S_1(t)dt \qquad (1)$$

And the risky asset with price dynamics, S_2 following the differential equation

$$dS_2(t) = \alpha\,(t)S_2(t)dt + \sigma S_2(t)dW(t) \qquad (2)$$

Where, W the standard Brownian motion, r is the rate of interest, α is the expected return on the risky asset, and σ is the volatility of the risky asset.

Let $r(t)_{short}$ denote the observed short-term interest rate.

The Mean equation is specified as:

$$r(t)_{short} = \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j Z_{st-j} + Z_{st} \qquad (3)$$

Where, Z_{St} is the innovation process, θ_j is the MA (moving average) coefficients, capturing the influence of past shocks on the current rate, ϕ_i is the AR (autoregressive) coefficients, capturing the influence of past rates on the current rate,

2.1 The EGARCH model for the Short-term interest rate

The exponential general autoregressive conditional heteroskedastic (EGARCH) is another form of the GARCH model. The EGARCH model was proposed by Nelson (1991) to overcome the weakness in GARCH's handling of financial time series. In particular, to allow for asymmetric effects between positive and negative asset returns.

The EGARCH (p, q) model has the variance model as

$$\log(\sigma(t)_s) = \omega_s + \sum_{i=1}^{p} \alpha_{si} \frac{Z_{st-i}}{\sqrt{\sigma(t-i)_s}} + \sum_{j=1}^{q} \beta_{sj} \log(\sigma\,(t-j)_s) + \sum_{k=1}^{p} \gamma_{sk} \left| \frac{Z_{St-k}}{\sqrt{\sigma(t-k)_s}} \right| \quad (4)$$

Where, $\sigma(t)_s$ denotes a volatility process, ω_s is the constant term, controlling the baseline level of volatility, α_{si} 's are the parameters capturing the impact of past standardized returns on volatility (asymmetric shock terms), β_{sj} 's arethe parameters representing the persistence of volatility and γ_{sk} 's terms allowing for asymmetric effects, where negative shocks can impact volatility differently than positive shocks, where subscript s for short-term interest rates.

2.2 The GJR-GARCH model for the Short-term interest rate

The Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model assumes a specific parametric form for this conditional heteroskedasticity.

The mean Equation of GJR-GARCH is same as specified in equation (3).

The GJR-GARCH (p, q), has the variance model as $\sigma(t)_S = \omega_s + \sum_{i=1}^{p} \alpha_{si} Z_{st-i}^2 + \sum_{j=1}^{q} \beta_{Sj} \sigma(t)_{s-j} + \sum_{i=1}^{p} \gamma_{si} Z_{st-i}^2 I$ (5)

The parameters have usual meaning as in the case of EGARCH model.

2.3 Optimal wealth after retirement

The proportion invested in the risky asset at time t is denoted by $u(t)$ and $(1-u(t))$ is the proportion in the riskless asset. The problem is to find optimal solution of $u(t)$.

State variable: The asset of pension plan is chosen as a state variable

$F(t)\ (t \in [N, N+T]$

Decision variable: Following the classical model of Merton (1971) the proportion invested in risky asset is chosen as the decision variable.

We model the wealth dynamics $F(t)$, incorporating stochastic interest rate, $r(t)$, over time t for the post-retirement phase. The wealth $F(t)$ is allocated between a risky asset and a risk-free asset, with the following dynamics,

$$dF(t) = F(t)\left[u(t)\alpha + (1-u(t))r(t)\right]dt + F(t)u(t)\sigma dW(t) \quad (6)$$

Where, $r(t)$is the stochastic interest rate on the risk-free asset.

We optimize the utility of the final surplus after retirement period with liabilities. Denoting C the part of the fund used to purchase annuity of periods and the surplus at the end of the fixed period can be used again in a similar way or paid back to the participants. The continuous benefit to pay between N and $N+T$ is given by

$$B = \frac{C}{1-e^{-\delta T}} \quad (7)$$

Where, $\boldsymbol{\delta}$ is the continuous technical rate

Objective function: The problem in two periods will be to optimize the expected utility of the final wealth at the end of the period. The maximization of the expected utility of the surplus after payment of pension during T periods is

$$\max_{u} EU\big(F(N+T)\big) \quad (8)$$

2.3.1 Optimal policy after retirement

Therefore, the final wealth equation after retirement for stochastic interest rate is as follows (Devolder et al., 2003), $(N \le t \le N+T)$,

$$dF(t) = F(t)\left[u(t)\alpha + (1-u(t))r(t)_{short} - B\right]dt + F(t)u(t)\sigma dW(t) \quad (9)$$

Using the classical tools of stochastic optimal control, to solve equation (8) with $F(N)$ the amount obtained at retirement.

Define the value function of the problem

$$V(t,F,r) = \max_{u} E[U(F(N+T)|F(t) = F] \quad (10)$$

The maximum principle leads to the following result (Hamilton- Jacobi method):

$$0 = \max_{\{u\}}[\frac{\partial V}{\partial t} + [[u(t)(\alpha - r(t)_{short}) + r(t)_{short}]F - B]\frac{\partial V}{\partial F} + \frac{1}{2}u^2(t)\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} + \mu_{r_s}\frac{\partial V}{\partial r_s} + \sigma_{r_s}^2 \frac{\partial^2 V}{\partial r_s^2}] \quad (11)$$

Or $0 = \max_{u}\{\phi\}$

We can derive from these two equations and second order condition:

(1). $\phi(u^*) = 0$ (12)

(2). $\frac{\partial \phi(u^*)}{\partial u} = 0$ (13)

(3). $\frac{\partial^2 \phi(u^*)}{\partial u^2} < 0$

Therefore (13) gives the optimal investment proportion u^* in risky asset and it is

$$u^*(t) = -\frac{{\partial V}/{\partial F}}{\left({\partial^2 V}/{\partial^2 F}\right)F} \quad (14)$$

Substituting this in equation (12), the partial differential equation for the value function is obtained as:

$$\frac{\partial V}{\partial t} + (r(t)_{short}F - B)\frac{\partial V}{\partial F} - \frac{1}{2}\frac{(\alpha - r(t)_{short})^2}{\sigma^2}\frac{\left(\partial V/_{\partial F}\right)^2}{\left(\partial^2 V/_{\partial^2 F}\right)} + \mu_{r_s}\frac{\partial V}{\partial r_s} + \sigma_{r_s}^2\frac{\partial^2 V}{\partial r_s^2} = 0 \qquad (15)$$

with limit condition $V(N+T, F, r) = U(F)$.
Solving equation (15) for the value function V and replacing it in (14), we obtain the optimal policy.

III.RESULTS AND DISCUSSION

The analysis uses daily data from Federal Reserve Economic Data (FRED) database, for the period January 1, 2022, to December 1, 2022. The DGS1MO rates were chosen to model short-term interest rates due to its relevance in investment strategies after retirement.

3.1. Overview of the datasets for Short-term interest rate

Pension payouts are often tied to short-term investments or annuities, where liquidity is important, and short-term rates provide a realistic reflection of what can be earned in liquid, low-risk investments like Treasury bills. Once in retirement, pension payouts are typically made periodically (e.g., monthly or annually). Therefore, a short-term rate like DGS1MO (1-Month U.S. Treasury Rates) is considered for the study.

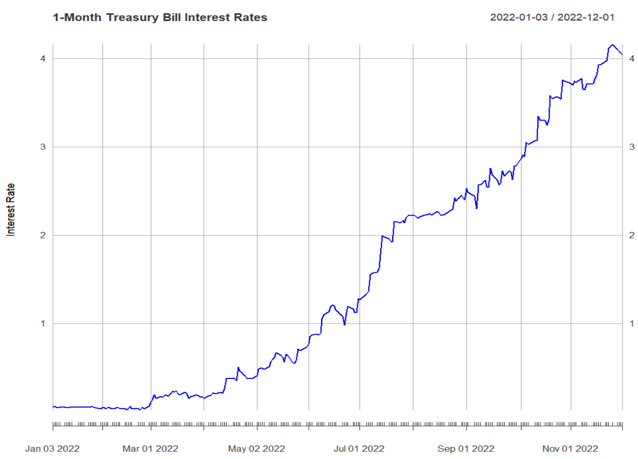


Figure 1. The plot of short-term interest rate (DGS1MO) from 1 Jan 2022 -1-Dec 2022

The above line plot shows the daily movement of DGS1MO.The DGS1MO series exhibit a clear upward trend over the year. Starting near zero in January 2022, the rates progressively increase, reaching approximately 4% by December 2022.The rise in rates appears to be relatively smooth, with some minor fluctuations. The ACF and PACF plot for the observed data and the output of Augmented Dickey-Fuller test.

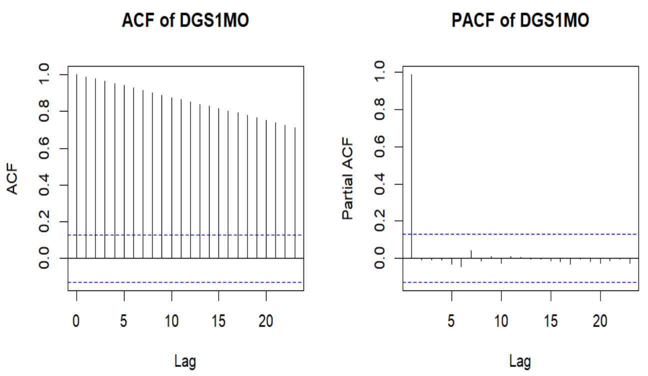


Figure 2. The ACF and PACF plot of Short-term interest rate (DGS1MO).

The ACF plot on the left show's significant autocorrelation at multiple lags. The PACF plot on the right shows a significant spike at lag 1 and smaller spikes at higher lags, indicating that the series might be autoregressive to some extent. The slow decaying of ACF indicate the presence of volatility clustering. The value of Dickey-Fuller Test for Short-term interest rate DGS1MO is -2.2829 with lag order 6, and the p-value is 0.4571. Since the p-value >0.05 we fail reject the hypothesis of non-stationarity hence the data is non stationary and implies no mean reversion. Therefore, we proceed with the GARCH type models by transforming the data for stationarity to model the stochastic interest rates.

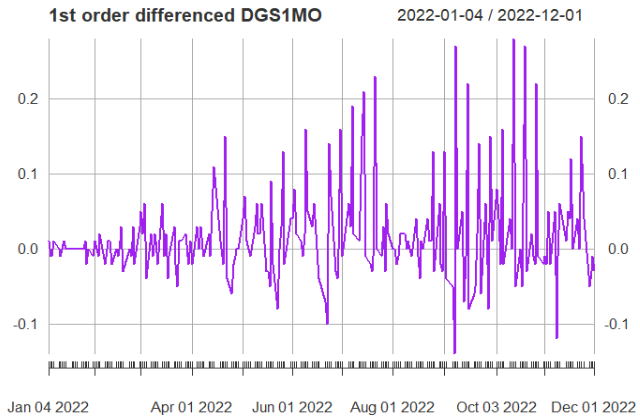


Figure 3. Plot of first order difference short-term interest rates (DGS1MO)

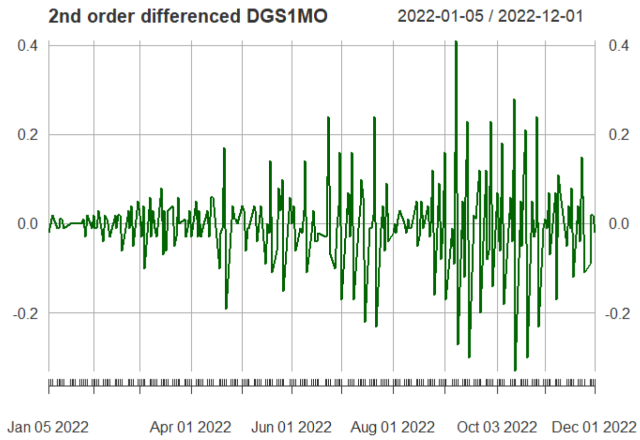


Figure 4. Plot of second order differenced short-term interest rates (DGS1MO)

The plots provide insights into the stationarity and behaviour of the short-term interest rate series (DGS1MO) after differencing the first-order and the second-order. The second-order differenced DGS1MO (Figure 4) also fluctuates around zero and appears to have reduced long-term trends compared to the first-order differenced series. This might imply that a GARCH-type model would be appropriate for capturing the dynamics of the series.

3.2. GARCH Model Estimation

The models used to analyze the data are EGARCH and GJR-GARCH type for conditional variance dynamics with an Auto Regressive Fractional Integrated Moving Average (ARFIMA) mean structure.

3.2.1. Estimation of the Model Parameters

The parameters were estimated using the maximum likelihood method under a generalized error distribution assumption. The results are summarized in Table 1and Table 3.

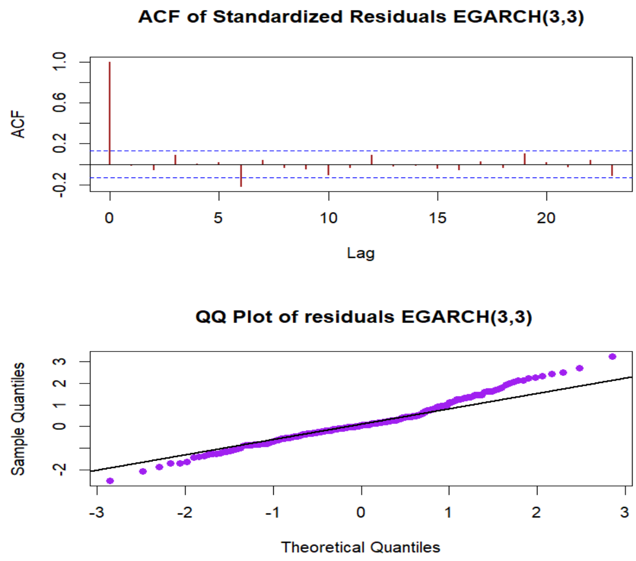


Figure 5. ACF and QQ plot of Standardized Residuals of EGARCH

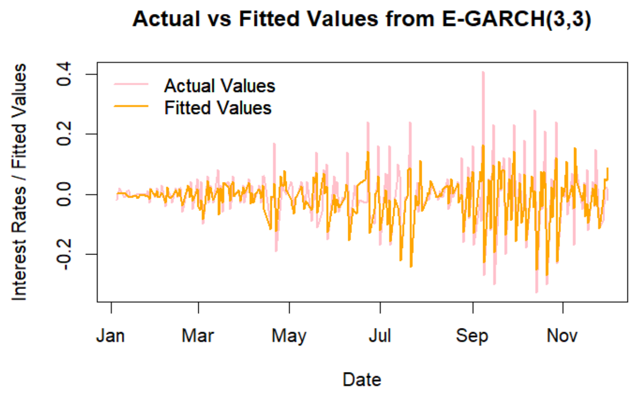


Figure 6. Plot of actual verses fitted values from EGARCH

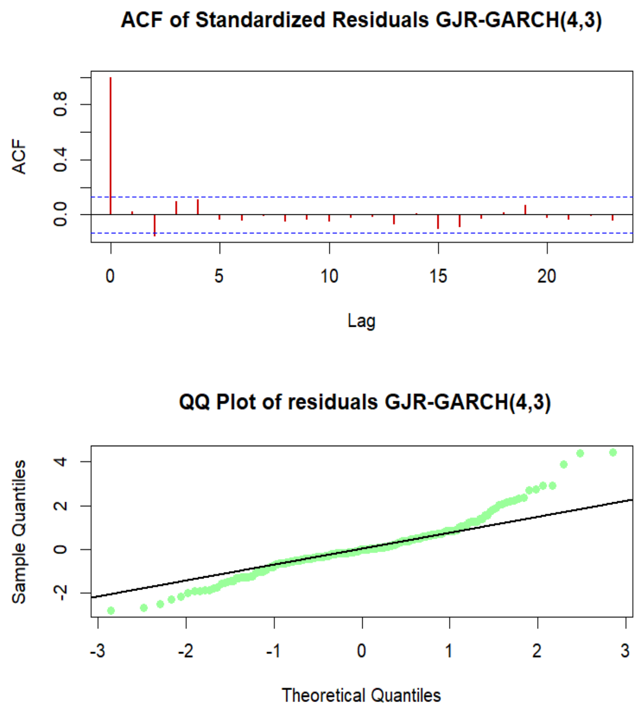


Figure 7.ACF and QQ plot of Standardized Residuals of GJR-GARCH

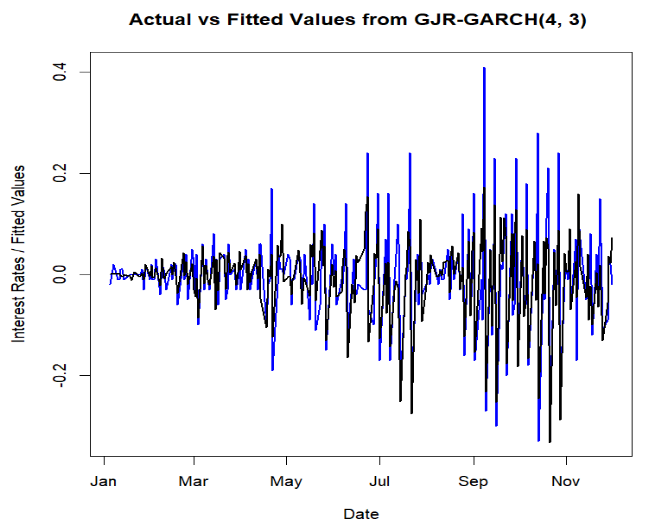


Figure 8. Plot of actual verses fitted values from GJR-GARCH

Table 1. Parameter Estimates for EGARCH(3, 3) with ARFIMA(5, 0, 4).

Parameters	Estimate	Std. Error	t-value	p-value
Mean Model Parameters				
μ_s	0.000256	0.000000	1071.0729	0.000000
AR(1)	0.008665	0.001215	7.1289	0.000000
AR(2)	-0.201029	0.000233	-862.8699	0.000000
AR(3)	-0.200782	0.000595	-337.3244	0.000000
AR(4)	-0.013484	0.003687	-3.6571	0.000255
AR(5)	0.245870	0.000467	526.2825	0.000000
MA(1)	-1.111595	0.000332	-3343.7337	0.000000
MA(2)	0.316763	0.000159	1987.3065	0.000000
MA(3)	-0.258012	0.000120	-2148.0948	0.000000
MA(4)	0.037595	0.000038	980.7524	0.000000
Variance Model Parameters				
ω_s	-0.586218	0.002428	-241.4173	0.000000
α_{s1}	-0.271071	0.001917	-141.4092	0.000000
α_{s2}	0.285118	0.027205	10.4804	0.000000
α_{s3}	-0.153100	0.007349	-20.8341	
β_{s1}	-0.060621	0.003423	-17.7107	0.000000
β_{s2}	0.168792	0.000327	515.7613	0.000000
β_{s3}	0.784461	0.000718	1091.9931	0.000000

Parameters	Estimate	Std. Error	t-value	p-value
γ_{s1}	1.351888	0.002313	584.3523	0.000000
γ_{s2}	0.335016	0.002394	139.9522	0.000000
γ_{s3}	-0.113289	0.009576	-11.8302	0.000000
GED distribution parameter				
shape	1.092168	0.034048	32.0772	0.000000

Table 2. Model Selection Criteria EGARCH Models.

Model	Loglikelihood	AIC	BIC
EGARCH(2, 3)	390.734	-3.2752	-2.9885
EGARCH(3, 3)	395.7899	-3.3021	-2.9853
EGARCH(3, 4)	388.5834	-3.2298	-2.8979
EGARCH(4, 4)	393.3356	-3.2541	-2.8919

Table 3. Parameter Estimates for GJR-GARCH(4, 3) with ARFIMA(5, 0, 4).

Parameters	Estimate	Std. Error	t-value	p-value
Mean Model Parameters				
μ_s	0.000136	0.000003	49.4585	0.000000
AR(1)	-0.329068	0.006378	-51.5942	0.000000
AR(2)	0.158684	0.004632	34.2566	0.000000
AR(3)	-0.215016	0.002154	-99.8306	0.000000
AR(4)	-0.081443	0.000554	-147.1142	0.000000
AR(5)	0.236886	0.002937	80.6651	0.000000
MA(1)	-0.816330	0.001046	-780.7199	0.000000
MA(2)	-0.363125	0.001127	-322.1353	0.000000
MA(3)	0.181358	0.001329	136.4589	0.000000
MA(4)	-0.007489	0.000293	-25.5455	0.000000
Variance Model Parameters				
ω_s	0.000011	0.000005	2.2960	0.021674
α_{s1}	0.016163	0.001646	9.8196	0.000000
α_{s2}	0.014209	0.000277	51.3570	0.000000
α_{s3}	0.014006	0.000522	26.8476	0.000000
α_{s4}	0.016398	0.000600	27.3472	0.000000
β_{s1}	0.218073	0.001439	151.5348	0.000000
β_{s2}	0.214188	0.001508	141.9991	0.000000
β_{s3}	0.220751	0.001542	143.1908	0.000000
β_{s4}	0.187389	0.001369	136.9248	0.000000
γ_{s1}	0.322787	0.002574	125.4082	0.000000
γ_{s2}	-0.207970	0.001377	-151.0598	0.000000
γ_{s3}	0.222534	0.005875	37.8762	0.000000
γ_{s4}	-0.167833	0.001974	-85.0128	0.000000

Parameters	Estimate	Std. Error	t-value	p-value
GED distribution parameter				
shape	0.880599	0.098144	8.9725	0.000000

Table 4. Model Selection Criteria of GJR-GARCH Models.

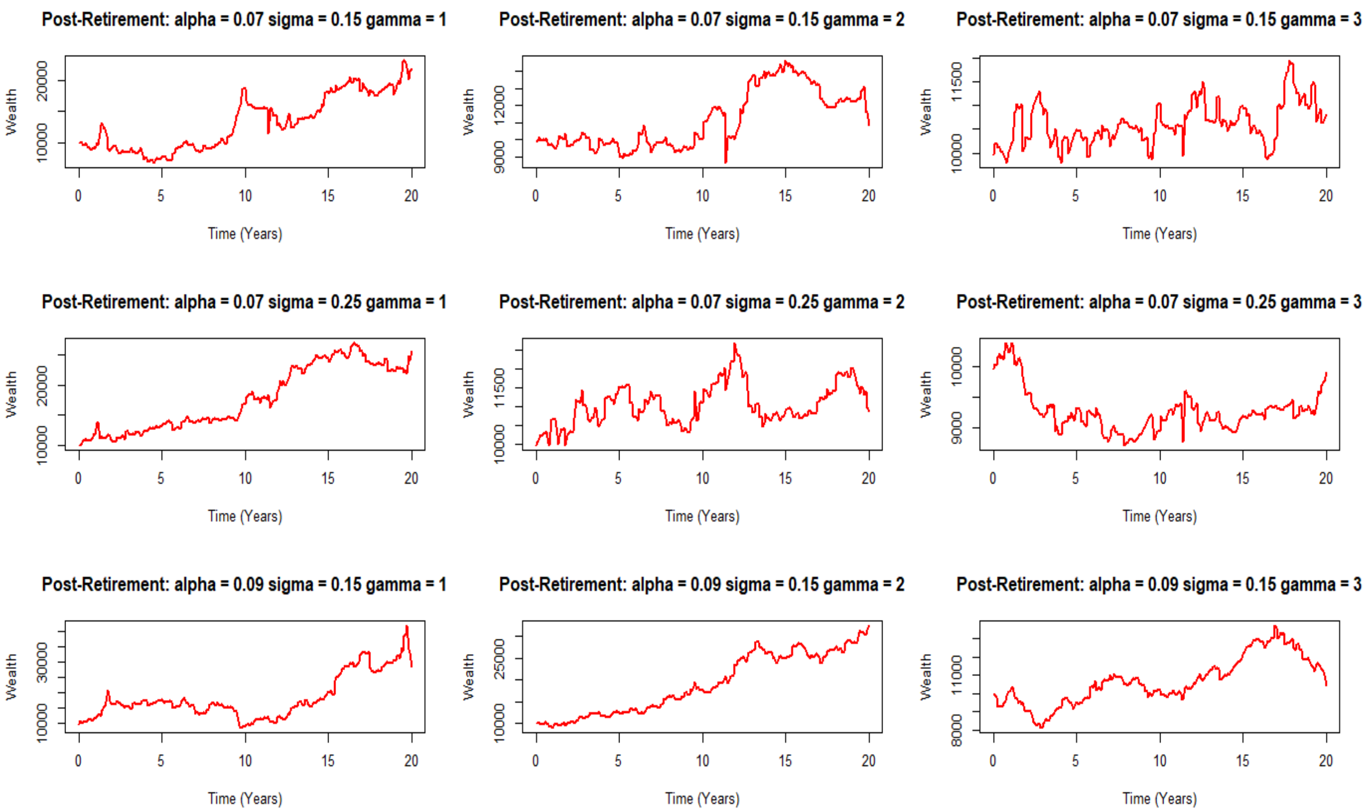
Model	Loglikelihood	AIC	BIC
GJR-GARCH(3,3)	374.7491	-3.1167	-2.7999
GJR-GARCH(3,4)	374.4659	-3.1054	-2.7735
GJR-GARCH(4,3)	378.7684	-3.1345	-2.7875
GJR-GARCH(4,4)	373.9528	-3.0833	-2.7212

It is observed from the Table 1, that the parameter estimates of EGARCH(3,3) are significant and the loglikelihood (Table 2) is maximum for EGARCH(3,3) with the lowest AIC and BIC. Similarly in the case of GJR-GARCH(4,3), it is observed that the parameter estimates (Table 3) are significant, the loglikelihood (Table 4) is maximum for GJR-GARCH(4, 3) with the lowest AIC and BIC. Furthermore, the MSE and RMSE values provide additional support for model selection. The MSE and RMSE for EGARCH(3,3) are 0.8679 and 0.9316, respectively, whereas for GJR-GARCH(4,3), the MSE and RMSE are 1.1957 and 1.0935, respectively. The lower MSE and RMSE values of the EGARCH(3,3) model indicate better predictive accuracy compared to GJR-GARCH(4,3). Additionally The ACF and QQ plot (*Figure 5*) implies that most of the lags fall within the confidence band suggesting no autocorrelation in the residual and they are normally distributed. The actual versus fitted value plot (*Figure 6)* also shows that the model fits most of the observed values and also indicates that the model has captured the volatility of the short-term interest rate. The same is observed in the case GJR-GARCH (4, 3). From the above observation it implies that EGARCH(3, 3) model fits the short-term interest rate better than GJR-GARCH (4, 3) model, based on both information criteria (AIC, BIC) and error measures (MSE, RMSE). Hence in the study EGARCH(3,3) model is chosen to model short-term interest rate for further analysis.

3.3. Sensitivity Analysis for short-term interest rate

Each subplot of Figure 9 represents a different combination of the parameters, allowing us to analyze how they impact post-retirement wealth sustainability over 20 years.

Table 5. Sensitivity Analysis for Post- Retirement Wealth Trajectories



Key Observations from the above figure

- Effect of Expected Return (α): Higher α (0.09) leads to better wealth accumulation, as seen in the bottom row. Lower α (0.07) results in more moderate growth, making it critical to balance return expectations with risk aversion.
- Effect of Volatility (σ): Higher volatility (σ = 0.25) increases fluctuations, making wealth unpredictable. Lower volatility (σ = 0.15) results in smoother wealth paths, reducing downside risk but also limiting potential gains.
- Effect of Risk Aversion: Low risk aversion (γ = 1)): Wealth trajectories show more aggressive growth but also higher. fluctuations. Some paths indicate significant gains, while others exhibit substantial drawdowns.
- Moderate risk aversion (γ = 2): Wealth is more stable compared to γ = 1. While growth is present, returns appear more conservative.
- High risk aversion (γ = 3): Wealth is less volatile but has slower growth.

Thus, the post-retirement wealth dynamics illustrates the significant impact of short-term interest rate volatility under the EGARCH model. The findings suggest that optimal policy requires cautious yet dynamic asset allocation strategy, ensuring risk minimization while sustaining wealth to meet retirement liabilities.

IV. CONCLUSION

This Study demonstrates that a moderate risk aversion strategy (γ = 2) with controlled exposure to market volatility (σ between 0.15-0.25) and a return rate of α = 0.07 or higher appears to be the most effective approach for balancing growth and sustainability in a defined contribution pension plan. Overly conservative strategies risk premature wealth depletion, whereas highly aggressive approaches expose retirees to significant financial losses.

By incorporating stochastic interest rate modeling, we show that EGARCH(3,3) and GJR-GARCH(4,3) effectively capture interest rate volatility, leading to improved asset allocation decisions. The application of the Hamilton-Jacobi-Bellman (HJB) equation further enables dynamic investment adjustments, allowing retirees to transition from high-risk to low-risk assets over time. Sensitivity analysis confirms that higher risk aversion levels lead to more conservative investment choices, reinforcing the importance of tailoring pension strategies to individual risk preferences.

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