International Journal of Scientific Research in Science and Technology
Print ISSN: 2395-6011 | Online ISSN: 2395-602X (www.ijsrst.com)
doi : https://doi.org/10.32628/IJSRST229110

Study of Convex Spaces and Their Tensor Products

Amaresh Kumar[1], Dr. Md. Mushtaque Khan[2]
[1] Research Scholar, University Department of Physics, J. P. University, Chapra, Bihar, India
[2] Department of Mathematics, K. R. College, Gopalganj, J. P. University, Chapra, Bihar, India

Article Info
Volume 9, Issue 1
Page Number : 475-479
Publication Issue
January-February-2022
Article History
Accepted : 01 Jan 2022
Published : 20 Jan 2022

ABSTRACT

In this paper, we will make considerable use of the notion of a continuous bilinear map X × Y → Z where X, Y and Z are topological vector spaces.

Keywords: Convex Spaces, Topology, Tensor, Vector Spaces, Functional Analysis.

I. INTRODUCTION

We will make considerable use of the notion of a continuous bilinear map X × Y → Z where X, Y and Z are topological vector spaces. In particular, we will make use of the following:

2.1 Proposition

If X, Y and Z are topological vector spaces then any bilinear map $\phi : X \times Y \to Z$ is continuous if it is continuous at (0,0).

Proof. Let (x_o, y_o) be a point of $X \times Y$ and let W be a 0-neighbourhood in Z [85-93]. We choose a 0-neigbhourhood W_1 with $W_1 + W_1 + W_1 \subset W$. Since ϕ is continuous at (0,0), there are 0-neighbourhood $U \subset X$ and $V \subset Y$ such that $\phi(U \times V) \subset W_1$. If we choose $s, t \in (0,1)$ so that $x_0 \in sU, yo \in tV, sU \subset U$ and $tV \subset V$, then for $x \in x_o$ +sU and $y \in y_o + tV$ we have

$$\phi(x,y) - \phi(x_o, y_o) = \phi(x - x_o, y_o) + \phi(x - x_o, y - y_o) + \phi(x_o, y - y_o) \subset W_1 + W_1 + W_1 \subset W.$$

This proves that ϕ is continuous at (x_o, y_o).

The algebraic tensor product $X \otimes y : x \in X, y \in Y\}$ subject to the relations

$$(ax_1 + bx_2) \otimes y = a(x_1 \otimes y) + b(x_2 \otimes y), x_1, x_2 \in X, y \in Y, a, b \in \mathbb{C}$$

$$x \otimes (ay_1 + by_2) = a(x \otimes y_1) + b(x \otimes y_2), x \in X, y_1, y_2 \in Y, a, b \in \mathbb{C}$$

It follows that every element $u \in X \otimes y$ may be written as a finite sum $u \sum_{i=1}^{n} x_i \otimes y_i$.

The minimal number n of terms required in a representation of u as above is called the rank of u. If u is expressed as above using a minimal number of terms, that is, so that the number of terms n is equal to the rank of u, then it turns out that the sets $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ must both be linearly independent. Of course, this representation of u is far from being unique.

One easily shows that the tensor product is characterized by the following universal property:

2.2 Proposition

The map $\theta : X \times Y \rightarrow X \otimes Y$, defined by $\theta(x, y) = x \otimes y$, is a bilinear map with the property that any bilinear map $X \times Y \rightarrow Z$ to a vector space Z is the composition of θ with a unique linear map $\psi : X \otimes Y \rightarrow Z$.

If X and Y are locally convex topological vector spaces, there are at least two interesting and useful ways of giving $X \otimes Y$ a corresponding locally convex topology [1-8]. The most natural of these is the projective tensor product topology, which we describe below:

If p and q are continous seminorms on X and Y, respectively, we define the tensor product seminorm $p \otimes q$ onn $X \otimes Y$ as follows:

$$(p \otimes q)(u) = \inf\left\{\sum p(x_i)q(y_i) : u = \sum x_i \otimes y_i\right\}$$

It follows easily that $p \otimes q$ is, indeed, a seminorm. Furthermore, we have:

2.3 Lemma

For seminorms p and q on X and Y.

(1) $(p \otimes q)(x \otimes y) = p(x)q \; for \; all \; x \in X, y \in Y;$

(2) If U = $\{x \in X : p(x) < 1\} \; and \; V = \{y \in Y : q(y) < 1\}, then$

$$co\left(\theta\,(U \times V)\right) = \{u \in X \otimes Y : (p \otimes q)(u) < 1\}$$

Proof. From the definition, it is clear that

$(p \otimes q)(x \otimes y) \leq p(x)q(y)$ for all $x \in X, y \in Y$

On the other hand, for a fixed $(x, y) \in X \times Y$, using the Hahn-Banach theorem we may choose linear functional f on X and g on Y such that $f(x) = p(x), g(y) = q(y)$ and $|f(x')| \leq p(x'), |q(y')$ for all $x', y') \in X \times Y$. Then if $x \otimes y = \sum x,\otimes y_i$ is any representation of $x \otimes y$ as a sum of rank one tensors, we have

$$p(x)q(y) = f(x)g)y) = \sum f(x_i)g(y_i) \leq \sum p(x_i)q(y_i)$$

Since $(p \otimes q)(x \otimes y)$ is the inf of the expressions on the right side of this inequality we have $p(x)q(y) \leq (p \otimes q)(x \otimes y)$. The proves (4.1).

Certainly $co\left(\theta(U \times V)\right) \subset \{u \in X \otimes Y : (p \otimes q)(u) < 1\}$ since the latter is a convex set containing $\theta(U \times V)$. To prove the reverse containment, let u be an element of $X \otimes Y$ with (p$\otimes q)(u) < 1$. Then we can represent u as $u = \sum x_i \otimes y_i$ with

$$\sum p(x_i)q(y_i) = r^2 < 1$$

If we set $x_i' = rp(x_i)^{-1}x_i \; and \; y_i' = rq(y_i)^{-1}y_i$, then $p(x_i') = r = q(y_i')$. Thus, $x_i' \in U$ and $y_i' \in V$. Furthermore, if $t_i = r^{-2}p(x_i)q(y_i)$. then

$u = \sum t_i(x_i' \otimes y_i')$ and $\sum t_i = 1$

Thus, $u \in co(\theta(U \times V))$ and the proof of (2) is complete.

3.1 Definition

The topology on $X \otimes Y$ determined by the family of seminorms $p \otimes q$, as above, will be called the projective tensor product topology. We will denote $X \otimes Y$, endowed with this topology, by $X \otimes Y$.

If $f \in X^*$ and $g \in Y^*$ then we may define a linear functional $f \otimes g$ on $X \otimes Y$ by

$$(f \otimes g)\left(\sum x_i \otimes y_i\right) = \sum f(x_i)g(y_i)$$

One easily checks that this is well defined and linear.

3.2. Proposition

The projective tensor product topology is a Hausfdorff locally convex topology on $X \otimes Y$ with the following properties [9-12] :

(1) the bilinear map $\theta : X \times Y \rightarrow X \otimes_\pi Y$ is continuous;

(2) $f \otimes g \in (X \otimes_\pi Y)^*$ for each $f \in X^*$ and $g \in Y^*$;

(3) A neighbourhood base for the topology at 0 in $X \otimes_\pi Y$ consists of sets of the form $co(\theta(U \times v))$ where U is a 0- neigbhourhood in X and V is a 0 neighbourhood in Y.

(4) any continuous bilinear map $X \times Y \to Z$ to a locally convex space Z factors as the composition of θ with a unique continuous linear map $X \otimes_\pi Y \to Z$;

Proof. Lemma 4.3(1) implies that each $(p\otimes q)o\,\theta$ is continuous at (0,0) and this implies that θ is continuous at (0,0) and, hence, is continuous everywhere by Proposition 4.1 [97-100].

The continuity of $f\otimes g$ for $f \in X^*$ follows from the fact that $|f|$ and $|g|$ are continuous seminorms on X and Y and $|(f\otimes g)(\sum x_i \otimes y_i)| \le \sum|f(x_i)||g(y_i)| \le \sum |f(x_i)||g(y_i)| \le |f| \otimes$|g|$(\sum x_i \otimes y_i)$This proves (4.2).

The fact that the projective topology is Hausdorff follows from (4.2). In fact, if $u \in X\otimes Y$ then we may write $u = \sum x_i \otimes y_i$, where the set $\{x_i\}$ is linearly independent. Then we may choose $f \in X^*$ such that $f(x_i) \ne 0$ if and only if $i = 1$ and we may choose $g \in Y^*$ such that $g(x_i) \ne 0$. Then the element $f\otimes g \in (X\otimes Y)^*$ has the non-zero value $f(x_1)g(x_1)$ at u. Thus, $U = \{v \in X\otimes Y : |(f\otimes g)(v)| < f(x_1)g(x_1)\}$ is an open set containing 0 but not containing u.

Part (3) is an immediate consequence of Lemma 4.3.(2)

If $\phi : X \times Y \to Z$ is a continuous bilinear map, then $\phi = \psi o\theta$ for a unique linear map $\psi : X \otimes_\pi Y \to Z$ by Proposition 4.2. To prove (4.4) we must show that ψ is continuous. Let W be a convex 0-neighbourhood in Z. Since ϕ is continuous, there exist 0-neighbourhoods U and V in X and Y, respectively, such that $\phi(U \times V) \subset W$. Then the convex hull of $\theta(U \times V)$ is a 0-neigbhourhood in $X\otimes Y$ by (4.3) and it clearly maps into W under ψ. Thus ψ is continuous.

Note that (4.4) of the proposition says that projective tensor product topology is the strongest locally convex topology on $X\otimes Y$ for which the bilinear map $\theta : X \times Y \to X \otimes Y$ is continuous.

Note that (4.4) of the above proposition says that projective tensor product topology is the strongest locally convex topology on $X\otimes Y$ for which determines its topology. Also, if X and Y are metrizable then so is $X\otimes Y$.

If X, Y and Z are locally convex spaces andn $\alpha : X \to Y$ is a continuous linear map then the composition

$$X \times Z \to Y \times Z \to Y \ \otimes_\pi Z$$

is a continuous bilinear map $X \times Z \to Y \times Z \to Y \ \otimes_\pi Z$ and, by Proposition 4.5.(4), it factors through a unique continuous linear map $\alpha\otimes id : X \otimes_\pi Z \to Y \otimes_\pi Z$. This shows that, for a fixed l.c.s. Z, $(.)\otimes_\pi Z$ is a function from the category of locally convex spaces to itself. Similarly, the projective tensor product is also a function in its second argument for each fixed l.c.s. appearing in its first argument.

3.3. Proposition

If $\alpha : X \to Y$ is a continuous linear open map, then so is $\alpha \otimes id : X \times Z \to Y \otimes Z$

Proof. To show that $\alpha \otimes id$ is open we must that each 0- neighbourhood in $X \otimes Z$ maps to a 0-neighbourhood in $Y \otimes Z$. However, this follows immediately from Proposition 4.5(3) and the hypothesis that α is an open map.

The space $X \otimes_\pi Y$ is generally not complete. It is usually useful to complete it.

4.1 Definition

The completion of $X \otimes_\pi Y$ will be denoted $X \ \widehat{\otimes}_\pi Y$ and will be called the completed projective tensor product of X and Y [13-15].

Note that if $\alpha : X \to Y$ is a continuous linear map, the map $\alpha \otimes id : X \widehat{\otimes}_\pi \to Y \ \widehat{\otimes}_\pi Z$. Even under the hypothesis of Proposition 4.6 this map is not generally a surjection. However, we do have:

4.2. Proposition

If X, X and Z are Frechet spaces and $\alpha : X \to Y$ is a surjective continuous linear map, then $\alpha \otimes id : X \widehat{\otimes}_\pi \to Y \ \widehat{\otimes}_\pi Z$.

Proof. By the open mapping theorem, the map a is open. Then $\alpha \otimes id$ is open by Proposition 3.6. Since, the topologies of X, Y and Z countable bases at 0 the same in true of $X \otimes_\pi Z$ and $Y \otimes_\pi Z$. However, an open map between metrizable t.v.s.'s has the property that every Cauchy sequence in the range has a subsequence which is the image of a Cauchy sequence in the domain. Since every point in the completion $X \widehat{\otimes}_\pi Z$ is the limit of a Cauchy sequence in $Y \widehat{\otimes}_\pi Z$, the result follows.

Obviously, the aualogues of Proposition 4.6 and 4.8 with the roles of the left and right arguments reversed are also true. There is other hypothesis under which the conclusion of the above Proposition is true and we will return to this question when we have developed the tools to prove such results [101-104]. In the case where X and Y are Fechet spaces, elements of the completed projective tensor product $X \widehat{\otimes}_\pi Y$ may be represented in a particularly useful form:

4.3. Proposition

If X and Y are Frechet spaces then each element $u \in X \widehat{\otimes}_\pi Y$ may be represented as the sum of a convergent series.

$$u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$$

$where \sum_{i=1}^{\infty} |\lambda_i| < \infty \ and$ $\{x_i\}$ and $\{y_i\}$ are sequences converging to 0 in X and Y, respectively.

Proof. Let $\{p_n\}$ and $\{q_n\}$ denote increasing sequences of seminorms generating the topologies of $p_n \otimes q_n$. If $\{u_n\}$ is a sequence in $X \otimes Y$ converging to u in the projective topology, then we may, by replacing $\{u_n\}$ by an appropriately chosen subsequence, assume that the sequence $\{u_n\}$, where $v_n = u_1$ and $v_n = u_n - u_{n-1}$ for $n > 1$, satisfies

$r_n(v_n) < n^{-2}2^{-n}$ and $\sum v_n = u$

It follows that we may write each v_n as a finite sum

$$v_n = \sum_i x'_{ni} \otimes y'_{ni} \ with \qquad \sum_i p_n(x'_i) q_n(y'_i) < n^{-2}2^{-n}$$

If we set

$$x_{ni} = n^{-1} p_a(x_{ni,}^{-1}) \ \ y_{ni} = n^{-1} p_n(y'_{ni})^{-1} y'_{ni}, \quad \lambda_{ni} = n^2 p_n(x_{ni}) q_n(y_{ni})$$

then

$$p_n(x_{ni}) = q_n(y_{ni}) = n^{-1}, \quad \sum_i |\lambda_{ni}| < 2^{-n} \ and \ \ v_n = \sum_i \lambda_{ni} x_i \otimes y_i$$

and so

$$u = \sum_{n,i} \lambda_{ni} x_i \otimes y_{i\,,} \sum_i |\lambda_{ni}| < \infty \ \ and \quad \lim_n pm(x_{ni}) = \lim_n pm(x_{ni}) = 0 \ \forall\, m$$

The proof is complete if we reindex $\{x_{ni}\}, \{y_{ni}\} \ and \ \{\lambda_{ni}\}$ to form singly indexed sequences.

REFERENCE

[1] Defant, A., The local Radon Nikod´ym property for duals of locally convex spaces, Bulletin de la Soci´et´e Royale des Sciencies de Li`ege, **53e** ann´ee, 5 (1984), 233-246.

[2] Dineen, S., Complex analysis on locally convex spaces, North-Holland Math. Stud., **57** (1981).

[3] Complex analysis on infinite dimensional spaces, Monog. Math., Springer- Verlag, 1999.

[4] Dineen, S., Galindo, P., Garc´ıa, D. and Maestre, M., Linearization of holomorphic mappings on fully nuclear spaces with a basis, Glasgow Math. J., **36** (1994), 201-208

[5] Galindo, P., Garc´ıa, D. and Maestre, M., Entire functions of bounded type on Fr´echet spaces, Math. Nachr., **161** (1993), 185-198.

[6] Horv´ath, J., Topological vector spaces and distributions, Vol. 1, Addison-Wesley, Massachusetts, 1966.

[7] Jarchow, H., Locally convex spaces, B.G. Teubner, 1981.

[8] K¨othe, G., Topological vector spaces I, Springer, London-Berlin-Heidelberg, 1969.

[9] Topological vector spaces II, Springer, London-Berlin-Heidelberg, 1979.

[10] Moraes, L., Holomorphic functions on strict inductive limits, Results Math., **4** (1981), 201-212.

[11] Venkova, M., Properties of Q-reflexive locally convex spaces, J. Korean Math. Soc., to appear.

[12] Q-reflexive locally convex spaces, Thesis, University College Dublin, 2002.

[13] R. Arens, Duality in linear spaces, Duke Math. J. vol. 14 (1947) pp. 787-794.

[14] N. Bourbaki, Éléments de mathématique. VII. Part 1, Les structures fondamentales de Vanalyse. Book II. Algèbre. Chapter III: Algèbre multilinêaire, Actualités Scientifiques et Industrielles, no. 1044, Paris, Hermann, 1948, 2+157+2 pp.

[15] Éléments de mathématique. X. Part 1. Les structures fondamentales de V analyse. Book III. Topologie générale. Chapter X. Espaces fonctionnels; dictionnaire, Actualités Scientifiques et Industrielles, no. 1084, Paris, Hermann, 1949. ii+101 pp.

Cite this Article

Amaresh Kumar, Dr. Md. Mushtaque Khan, "Study of Convex Spaces and Their Tensor Products", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 9 Issue 1, pp. 475-479, January-February 2022.

Journal URL : https://ijsrst.com/IJSRST22969