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Study of Duality of Locally Convex Space

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ABSTRACT

In this paper, we studied about the duality of locally convex space. The key to most of the results in topological vector space theory is to exploit duality - the relationship between on l.c.s. X and its dual X^*. The results of this section, particularly, show how this works. We will need to work with a variety of topologies on an l.c.s. X and it dual X^*. The results of this section, particularly, show how this works.

Keywords: Convex Spaces, Topology, Tensor, Vector Spaces, Functional Analysis.

I. INTRODUCTION

We work with a variety of topologies on an l.c.s. X and its dual X^*. Below we give a list of some of the more important topologies [1-7] on X and X^*.

1.1 Definition

Given an l.c.s. X and its dual X^* we define the following spaces, each of which is X or X^* with the indicated topology.

(1) X_σ −(the weak or $\sigma(X, X^*)$ topology) the topology of uniform convergence on finite subsets of X^*;

(2) X^*_σ- (the weak-* or $\sigma\ (X^*, X) topology$) the topology of uniform convergence on finite subsets of X :

(3) X_T − (the Mackey topology of X) the topology of uniform convergence on compact convex subsets of X^*_σ;

(4) X^*_T − (the Mackey topology of X) the topology of uniform convergence on compact convex subsets of X_σ;

(5) X − (the original topology on X) the topology of uniform convergence on equicontinuous subsets of X^*_σ;

The topology of uniform convergence on equicontinuous sets in X^* is the original topology of X due to the fact that a set is equicontinuous in X^* if and only if it is contained in V^o for V a 0-neighbourhood in X and the bipolar theorem, which implies that $V^{oo} = \bar{V}$ if V is convex and balanced [8-10].

Each topology in 1.1 is the topology of uniform convergence on sets belonging to a class S of subsets of a space in a dual pair relation with the space we are topologizing. Let us consider a dual pair of vector spaces (X, Y) and ask what properties of a family S of subsets of Y are needed in order that the topology of uniform convergence on members of S determines an l.c.s. topology on X. The topology of uniform converence on members of S is the topology generated by the family of seminorms of the form p_A for $A \in S$ where

$$q_A(x) = \sup\{|x(y)| = |< x, y >|: y \in A\}$$

Obviously, this supremum will not be finite for all $x \in X$ unless each function $x(y)$ for $x \in X$ is bounded on the set A. Thus, we need the sets in S to be bounded for the $\sigma(Y, X)$ topology. If every $y \in Y$ belongs to some $A \in S$ then each of the functions $y(x) =< x, y >$ will be dominated by one of the seminorms q_A and, hence, each $y \in Y$ will determine a continuous linear functional on X. Since the collection of such functions separates points, this will imply that X is Hausdorff in the topology generated by the seminorms q_A. If we want the family of seminorms to be directed, then for each finite set $\{A_i\} \subset S$ there should exist $B \in S$ such that $\cup A_i \subset B$. Under these conditions, the sets of the form a neigbourhood base at 0 for a Hausdorff

$$\{x \in X :\ q_A(x) < c\}\ A \in A, \qquad \in > 0$$

l.c.s. topology on X with the property that the elements of Y are all continuous as linear functional on X. If we also insist that $tA \in S$ whenever $A \in S$ and $t > 0$ then the polars A^o of sets $A \in S$ form a neigbourhood base at 0 for this topology. This discussion is summarized in the following proposition.

1.2 Proposition

Let (X, Y) be a dual pair and let S be a family of subsets of Y with the following properties.

(1) S is a family of $\sigma\ (Y, X)$ bounded subsets of Y;

(2) $\cup\{A : A \in S\} = Y$;

(3) the union of the members of any finite subset of S is contained in a member of S;

(4) If $A \in S$ and $t > 0$ then $tA \subset S$.

Then the topology of uniform convergence on members of S is a Hausdorff l.c.s. topology on X relative to which every $y \in Y$ determines a continuous linear functional. The polars in X of members of S form a neighbourhood base at 0 for this topology.

We will call a family S of subsets of Y that satisfies (1) - (4) of Prop. 5.2 a basic family. We will call the topology of uniform convergence on members of a basic family S the $S-$topology. Clearly the families determining the topologies described in Definition 5.1. are basic families.

By the bipolar theorem, a seet $A \subset Y$ and its closed, convex, balanced hull have the same polar in X. Therefore, a basic family S will determine the same topology as the basic family obtained by replacing each set in S by its $\sigma(Y, X)$-closed, convex, balanced hull. Thus, we could insist that the families S we use to define topologies consist of closed, convex, balanced sets. The reason we don't do this is that it is easier to say "uniform convergence on all finite sets" than to say "uniform convergence on all convex balanced hulls of finite sets". Never the less, wherever it is convenient, we shall assume that the sets in a basic family S are closed, convex and balanced.

1.3. Definition

Given a dual pair (X, Y), an l.c.s. topology on X is said to be consistent with the pairing (X,Y) provided the dual of X with this topology is equal to Y as a vector space.

Note this means that each function $y(x) =< x, y >$, for $y \in Y$ is continuous in the given topology on X and that every continuous linear functional has this form.

2. Theorem (Mackey-Arens)

A topology on X is consistent with the pairing (X, Y) if and only if it is the S $-$topology for a basic family of $\sigma(Y, X)$ compact, convex, balanced subsets of Y.

Proof. Let X be given a topology consistent with the pairing, so that $Y = X^*$, then the polar of each 0-neighbourdhood in X is a convex, blanced and $\sigma(X^*, X) -$ compact subset of X^* by the Banach-Aloaglu Theorem [119]. The bipolar theorem implies that the given topology on X is the $S -$topology for the basic family S consisting of all polars of 0-neighbourhoods in X. On the other hand, suppose S is a basic family of convex, balanced, $\sigma\ (X, Y)$ compact subsets of Y and give X the $S -$topology. Then for each $y \in Y$ the function $y(x) = < x, y >$ is a continuous linear functional on X. We must that every continuous linear functional on X arises in this way. Thus, suppose $f \in X^*$. Then $\{x \in X : |f(x| < 1\}$ is a 0- neghbourhood in X since f is continuous. It follows that this neighbourhood must contain a neighbourhood of the form $V = A^o$ for a set $A \in S$. Now $f \in V^o = A^{oo}$ where the bipolar is taken in X^* and A is considered a subset of X^* through the embedding $Y \to X^*$. Since A is $\sigma(Y, X)$ compact and $Y \to X^*$ is continuous from the $\sigma(Y, X)$ topology to the $\sigma\ (X^*, X)$ topology, A is closed in X^*. Now the bipolar theorem implies that $V^o = A$ and, hence, that $f \in A$. In particular, $f \in Y$. This completes the proof.

2.1. Corollary

Given a dual pair (X, Y) there is a strongest locally convex topology on X consistent with the pairing. It is the $r(X, Y) -$topology where $r(X, Y)$ is the basic family consisting of all convex, balanced $\sigma(Y, X)$- compact subsets of Y. This is called the Mackey topology on X for the pairing (X, Y). When the pairing is (X, X^*) for an l.c.s. the space X with the Mackey topology is the space X_* of Definition 5.1. When the pairing is (X^*, X) the space X^* with the Mackey topology is the space X_r^* of Definition 5.1.

Note that the $\sigma(X, Y)$ topology is the weakest locally convex topology consistent with the pairing (X, Y).

2.2. Definition

An l.c.s. X for which the original topology is the Mackey topology $r(X, X^*)$ is called a Mackey space.

2.3. Definition

If X is an l.c.s. then a barrel in X is a closed, convex, balanced absorbing set. If every barrel in X is a 0-neighbourhood, then X is said to be barreled.

Note Problem 1.1. which says that every Frechet space is barreled, Problem 1.2 which says that every quasi-complete bornological space is barreled and Problem 1.3 which says that every bornological space is a Mackey space and every barreled space is a Mackey space.

2.3. Proposition

An l.c.s. X is barreled if and only if every $\sigma(X^*, X) -$ bounded subset of X^* is equicontinuous.

Proof. A set $B \subset X^*$ is $\sigma(X^*, X) -$ bounded if and only if each $x \in X$ is bounded on B when considered a linear functional on X^*. This is the case if and only if each x belongs to tB_0 for some $t > 0$, that is, if and only if B_o is absorbing. In other words, the $\sigma(X^*, X)$- bounded subsets of X^* are those whose polars are absorbing. On the other hand, the equicontinuous subsets of X^* are those whose polar are 0-neighbourhoods. Obviously then, every barrel in X is a 0-neigbhourhood if and only if every $\sigma(X^*, X)$- bounded subset of X^* is equicontinuous.

Note that the pairing (X^*, X) expresses X as a space of linear functional on X^* and these are clearly continuous in any $S -$topology for a basic family S of bounded subsets of X. This is true, in particular, for the strong topology on X^*. Thus, there is a canonical linear map $X \to \left(X_\beta^*\right)^*$. If we also give X_β^*. If we also give $(X_\beta^*)^*$ its strong topology, then we have a canonical map $X \to (X_\beta^*)_\beta^*$. This is not, in general, even continuous. However, we have:

3. Proposition

The canonical map $X \to (X_\beta^*)_\beta^*$ is an open map into its range. It is a topological isomorphism onto its range if X is barreled [10].

Proof. A typical 0-neigbhourhood in $(X_\beta^*)_\beta^*$ is of the form $U = B^o$ for a bounded set $B \subset X_\beta^*$, also, $V = U \cap X$ is the polar, taken in X, of B. This will be a 0-neighbourhood in X exactly when B is equicontinuous. Equicontinuous sets in X^* are strongly bounded and, so the map $X \to (X_\beta^*)_\beta^*$ is always open onto its range. If X is barreled then $\sigma\,(X^*, X)$-bounded sets of X^* are equicontinuous and, hence, strongly bounded sets are equicontinuous. It follows that $X \to (X_\beta^*)_\beta^*$ is continuous if X is barreled. This completes the proof.

3.1. Definition

An l.c.s. X is called reflexive if the natural map $X \to (X_\beta^*)_\beta^*$ is a topological isomorphism.

3.2. Proposition

An l.c.s. X is reflexive if and only if it is barreled and every bounded subset of X has weakly compact weak closure.

Proof. If X is barreled then $X \to (X_\beta^*)_\beta^*$ is a topological isomorphism onto its range. If every bounded subset of X has weakly compact weak closure, then every closed, convex, balanced, bounded set is weakly compact and it follows that the strong and Mackey topologies agree on X^*. Thus, by the Mackey-Arens theorem, the dual of X_β^* is X. In other words, $X \to (X_\beta^*)_\beta^*$ is surjective.

To prove the converse, note first that if X is reflexive then the $\sigma(X^*, X)$ topology on X^* is the weak topology of the l.c.s. X_β^*, since this space has dual equal to X. Thus, $\sigma(X^*, X)$ bounded sets in X^* are weakly bounded and, hence, strongly bounded subsets of X_β^*. Also by reflexivity, a set in X_β^* is bounded if and only if its polar is a 0-neighbourhood in X. This implies that bounded sets in X_β^* are equicontinous and, hence, that $\sigma(X^*, X)$ bounded sets are equicontinuous. By Prop. 4,8, X is barreled. Now if X is reflexive then X_β^* and its strong dual is X. Thus, we also conclude that bounded subsets of X are equicontinuous as sets of functional on X_β^*. The Banach-Aloaglu theorem then implies that bounded sets have $\sigma(X^*, X)$ compact closures. This completes the proof.

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