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Study of Stability of Equilibrium Points in The Pcr3bp on the Circumference of FEC

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ABSTRACT
The Three Body Problem The three body problem studies the motion of three masses whose gravitational attraction have an effect on each other. The dynamics of the three-body problem are essentially different from those of two bodies, because in the latter case, an analytical solution may be found that admits orbits in the form of conic sections. This problem has been studied at great length and is the basis of most of today's orbit planning and trajectory design for satellites. However, the two-body problem is valid on close to a single massive body, compared to which the target body (the object whose motion is desired) is essentially a massless particle. In deep space, when there may be two or more massive bodies to affect the motion of our test particle, the two-body solution obviously fails. It then becomes essential to study the three-body problem.
Keywords : RTBP, CRTBP, Three Body Problem.

I. INTRODUCTION

In the restricted three body problem (RTBP), the target mass/test particle is assumed to be of negligible mass when compared to the other two bodies (called primaries). The two primaries orbit around their common center of mass and their motion is unaffected by the test particle. Common examples of the RTBP are: a satellite in the Earth-Moon system, asteroids in the Sun-Jupiter system, etc.

The most common alternative to the RTBP is the method of patched conics. Here, it is assumed that the test particle is only under the effect of one primary when in its vicinity, and under the effect of the other primary when close to it. This allows the solution in terms of two conic analytical solutions to the orbit, hence the name patched conics. Here we must introduce the concept of the sphere of influence - the sphere enclosing the second primary of comparatively less mass, in which the effects of the more massive primary can be "switched off". At the point of entry into the sphere of influence, the position and velocity conditions of the two conic orbits are matched to ensure continuity. Obviously, it is less accurate than the three-body problem and only serves as a preliminary trajectory design tool.

The RTBP has been the subject of constant study for the last few centuries. It has given rise to many original ideas (for example, Poincare's sections) and stimulated other branches of mathematics and mechanics like topology. A study of the RTBP can greatly enhance the knowledge and understanding of mathematical techniques presently used, since it has been the reason of the birth of the same. The RTBP is found to admit some equilibrium solutions (attributed to Euler and Lagrange). This report concerns the stability analysis of these points of equilibria using classical and modern mathematical techniques.

We know that an equilibrium point is one where the velocity (or derivative of the states) is zero. This is true of the system being considered, but only in the rotating frame. When we consider the inertial frame, these points will rotate in a circle about the barycenter, and hence they are called relative equilibria. For the rest of this report, equilibrium will be used synonymously with relative equilibrium.

II. THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

Consider a mechanical system consisting of three gravitationally interacting point masses, M_1, M_2, and m. Suppose, that the third mass, m, is so much smaller than the other two that it has a negligible effect on their motion. Suppose, further, that the first two masses, M_1 and M_2, execute a circular orbit about their common centre of mass. This simplified problem is known as the circular restricted three-body problem.

Let us further assume, to simplify the presentation of the final calculations, that mass m moves in the plane of the orbital motion of masses M_1 and M_2.

Let ω be the constant orbital angular velocity of masses M_1 and M_2 on the circular orbit. We can find ω by equating Fcp, the centripetal force acting upon the mass $\mu = \frac{M_1 M_2}{M_1+M_2}$(the equivalent one-body problem), and Fg, the force of gravitational attraction between masses M_1 and M_2:

$$F_{cp} = \frac{M_1 M_2}{M_1+M_2}\frac{v^2}{R'} \qquad F_g = G\frac{M_1 M_2}{R^2} \tag{1}$$

where G is the gravitational constant, v is the constant linear velocity of mass μ. From Eq. (1)

$$v^2 = G\frac{M_1+M_2}{R^2} \tag{2}$$

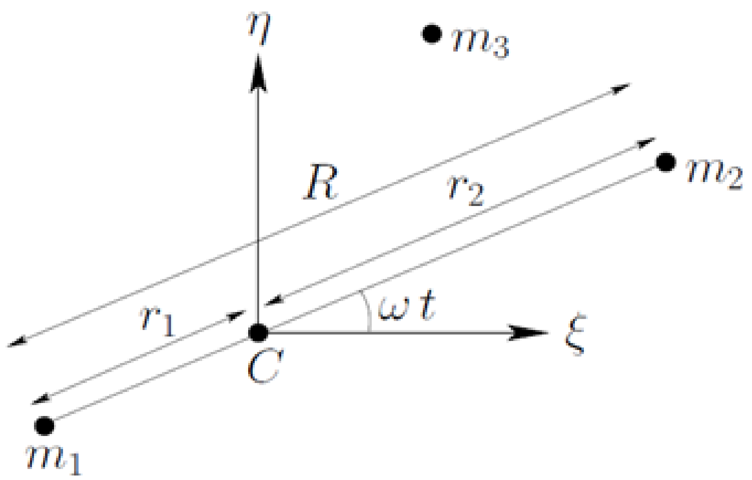

Figure 1: The circular restricted three-body problem

On the other hand, the period of orbital motion on a circular orbit, T, is

$$T = \frac{2\pi R}{v} \tag{3}$$

thus,

$$w = \frac{2\pi}{T} = \frac{v}{R} \tag{4}$$

Substituting Eq. (2) into Eq. (4) we arrive at the following expression.

$$w^2 = G\frac{M_1+M_2}{R^3} \tag{5}$$

Let us define a Cartesian coordinate system (ξ, η, ζ) in an inertial reference frame whose origin coincides with the center of mass, C, of the two orbiting masses, M1 and M2. Let the orbital plane of these masses coincide with the ξ-η plane, and let them both lie on the ξ-axis at time t = 0 — see Figure 5.1. Suppose that R is the constant distance between the two orbiting masses, r1 the constant distance between mass M1 and the origin, and r_2 the constant distance between mass M_2 and the origin.

Let the third mass have position vector $\vec{r} = (\xi, \eta, 0)$. The Cartesian components of the equation of motion of this mass are thus

$$\ddot{\xi} = -GM_1\frac{(\xi-\xi_1)}{\rho_1^3} - GM_2\frac{(\xi-\xi_2)}{\rho_2^3} \tag{6}$$

$$\ddot{\eta} = -GM_1\frac{(\eta-\eta_1)}{\rho_1^3} - GM_2\frac{(\eta-\eta_2)}{\rho_2^3} \tag{7}$$

$$\rho_1^2 = \left(\xi-\xi_1\right)^2 + (\eta-\eta_1)^2 \tag{8}$$

$$\rho_2^2 = \left(\xi-\xi_2\right)^2 + (\eta-\eta_2)^2 \tag{9}$$

3. CO-ROTATING FRAME

Let us transform to a non-inertial frame of reference rotating with angular velocity ω about an axis normal to the orbital plane of masses M_1 and M_2, and passing through their center of mass. The masses M_1 and M_2 are stationary in this new reference frame. Let us define a Cartesian coordinate system (X, Y) in the rotating frame of reference which is such that massesM$_1$ and M_2 always lie on the X-axis. Let the position vector of mass m be $\vec{r}$ = (x, y) see Figure 2.

The masses M1 and M2 have the fixed position vectors

$$\vec{r}_1 = (-\alpha R, 0,0) \qquad \vec{r}_2 = (1-\alpha)R, 0,0) \tag{10}$$

in our new coordinate system. Indeed, by the definition of the center of mass,

$$r_1M_1 = r_2\, M_2 \tag{11}$$

on the other hand

$$r_1 + r_2 = R. \tag{12}$$

Solve Eqs. (11) and (12), we obtain,

$$r_1 = \frac{M_2}{M_1+M_2}R,\ r_2 = \frac{M_1}{M_1+M_2}R = \left(1-\frac{M_2}{M_1+M_2}\right)R \tag{13}$$

i.e. in Eq. (10)

$$\alpha = \frac{M_2}{M_1+M_2} \tag{14}$$

The equation of motion of mass m in the reference frame are obtained by including into Eqs. (6), (7) two additional forces - Coriolis force $\vec{F}_{cor}$ and centrifugal force $\vec{F}_{cf}$:

$$\vec{F}_{cf} = -m\vec{w}\times(\vec{w}\times\vec{r}) = mw^2\vec{r}, \tag{15}$$

$$\vec{F}_{cor} = -2m\vec{w}\times\dot{\vec{r}} = 2mw(-\hat{x}\dot{y}+\hat{y}\dot{x}) = mw^2\vec{r}, \tag{16}$$

$$\ddot{\vec{r}} = -GM_1\frac{(r-r_1)}{\rho_1^3} - GM_2\frac{(\vec{r}-\vec{r}_2)}{\rho_2^3} - \vec{w}\times(\vec{w}\times r) - 2\vec{w}\times\dot{r} \tag{17}$$

where $\vec{w}$ = (0, 0, w), and

$$\rho_1^2 = (x+\alpha R)^2 + y^2, \tag{18}$$

$$\rho_2^2 = (x-(-\alpha)R)^2 + y^2, \tag{19}$$

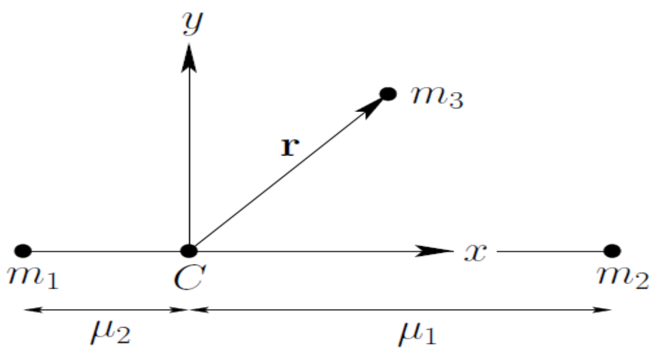

Figure 2 : The Co-rotating Frame

Here, the last two terms on the right-hand side of Eq. (17) are the centrifugal acceleration and the Coriolis acceleration.

The components of Eq. (17) reduce to

$$\ddot{x} = -\frac{GM_1(x+\alpha R)}{\rho_1^3} - \frac{GM_2(x-(1-\alpha)R)}{\rho_2^3} + w^2x + 2w\dot{y}, \tag{20}$$

$$\ddot{y} = -\frac{GM_1 y}{\rho_1^3} - \frac{GM_2 y}{\rho_2^3} + w^2x - 2w\dot{x} \tag{21}$$

III. JACOBI INTEGRAL

Eqs. (20), (21) can be rewritten as following.

$$\ddot{x} - 2w\dot{y} = -\frac{\partial U}{\partial x} \tag{22}$$

$$\ddot{y} - 2w\dot{x} = -\frac{\partial U}{\partial y} \tag{23}$$

where

$$U = -\frac{GM_1}{\rho_1} - \frac{GM_2}{\rho_2} - \frac{w^2}{2}(x^2 + y^2) \tag{24}$$

is the sum of the gravitational and centrifugal potentials

Now, it follows from Eqs. (22) - (23) that

$$\ddot{x}\dot{x} - 2w\,\dot{x}\,\dot{y} = -\dot{x}\frac{\partial U}{\partial x} \tag{25}$$

$$\ddot{y}\dot{y} - 2w\,\dot{x}\,\dot{y} = -\dot{y}\frac{\partial U}{\partial x} \tag{26}$$

Summing the above equations, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U\right] = 0 \tag{27}$$

In other words,

$$C = -2U - v^2 \tag{28}$$

is a constant of the motion, where $v^2 = \dot{x}^2 + \dot{y}^2$. C is called the Jacobi integral. The mass m is restricted to regions in which

$$-2U \geq C \tag{29}$$

since v^2 is a positive definite quantity.

IV. DIMENSIONLESS FORM OF THE EQUATIONS

No analytic solutions of Eqs. (20) - (21) are known. My goal is to solve them numerically. As the first required step, we convert the to a dimensionless form.

Circular restricted three body problem has natural scales: the distance R, between masses M_1 and M_2 and the characteristic time of their orbital motion 1/w. Let us introduce dimensionless variables by measuring the coordinates x and y in units of R, thus introducing new unknowns u and v as following.

$$u \equiv \frac{x}{R}, \qquad v \equiv \frac{y}{R} \tag{30}$$

Let us measure time t in units of 1/w, introducing dimensionless variable τ.

$$\tau \equiv \mathrm{wt} \tag{31}$$

"Old" derivatives with respect to time are going to have the following forms:

$$\dot{x} \equiv \frac{dx}{dt} = \frac{d(uR)}{d(\tau/\mathrm{w})} = wR\frac{du}{d\tau} \tag{32}$$

$$\ddot{x} \equiv \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\mathrm{wR}\frac{du}{d\tau}\right) = \mathrm{wR}\ \frac{d}{d\tau/\mathrm{w}}\left(\frac{du}{d\tau}\right) = \mathrm{w}^2\mathrm{R}\frac{d^2u}{d\tau^2}. \tag{33}$$

Similarly,

$$\dot{y} = wR\frac{dv}{d\tau} \tag{34}$$

$$\ddot{y} = w^2R\frac{d^2v}{d\tau^2} \tag{35}$$

Substituting Eqs. (32) - (35) into Eqs. (20), (21), we get

$$w^2R\frac{d^2u}{d\tau^2} = -\frac{GM_1R(u+\alpha)}{\rho_1^3} - \frac{GM_2R(u+\alpha)}{\rho_2^3} + w^2Ru + 2w^2R\frac{dv}{d\tau} \tag{36}$$

$$w^2R\frac{d^2v}{d\tau^2} = -\frac{GM_1Rv}{\rho_1^3} - \frac{GM_2Rv}{\rho_2^3} + w^2Rv - 2w^2R\frac{du}{d\tau} \tag{37}$$

Here and expressed via dimensionless parameters are as following:

$$\rho_1 = R((u+\alpha)^2 + v^2)^{\frac{1}{2}} = Rd_1 \tag{38}$$

$$\rho_2 = R((u-1+\alpha)^2 + v^2)^{\frac{1}{2}} = Rd_2 \tag{39}$$

where

$$d_1 = ((u+\alpha)^2 + v^2)^{\frac{1}{2}} \tag{40}$$

$$d_2 = ((u-1+\alpha)^2 + v^2)^{\frac{1}{2}} \tag{41}$$

Dividing term in Eqs. (36) - (37) by w^2R, we arrive the following equations.

$$\frac{d^2u}{d\tau^2} = -\frac{GM_1}{w^3R^3}\frac{(u+\alpha)}{d_1^3} - \frac{GM_2}{w^3R^3}\frac{(u-1+\alpha)}{d_2^3} + u + 2\frac{dv}{d\tau} \tag{42}$$

$$\frac{d^2v}{d\tau^2} = -\frac{GM_1}{w^3R^3}\frac{v}{d_1^3} - \frac{GM_2}{w^3R^3}\frac{v}{d_2^3} + v - 2\frac{du}{d\tau} \tag{43}$$

Noticing that

$$\frac{GM_1}{w^3R^3} = \frac{M_1}{M_1+M_2} \equiv 1-\alpha \tag{44}$$

and

$$\frac{GM_2}{w^3R^3} = \frac{M_1}{M_1+M_2} \equiv \alpha \tag{45}$$

we arrive the following equations

$$\frac{d^2u}{d\tau^2} = -(1-\alpha)\frac{(u+\alpha)}{d_1^3} - \alpha\frac{(u-1+\alpha)}{d_2^3} + u + 2\frac{dv}{d\tau} \tag{46}$$

$$\frac{d^2v}{d\tau^2} = -(1-\alpha)\frac{v}{d_1^3} - \alpha\frac{v}{d_2^3} + v - 2\frac{du}{d\tau} \tag{47}$$

Equations (5.46) - (5.47) can be rewritten in a compact form

$$\ddot{u} = -\frac{\partial U}{\partial v} + 2\dot{v}, \tag{48}$$

$$\ddot{v} = -\frac{\partial U}{\partial v} + 2\dot{u} \tag{49}$$

where

$$U(u,v) = -\frac{1-\alpha}{d_1} - \frac{\alpha}{d_2} - \frac{1}{2}(u^2+v^2) \tag{50}$$

is the dimensionless version of Eq. (5.24).

Equation (46) - (47) are dimensions and contain a single parameter, α. Some of the results of their numerical solution are presented in Figs. 3 and 4.

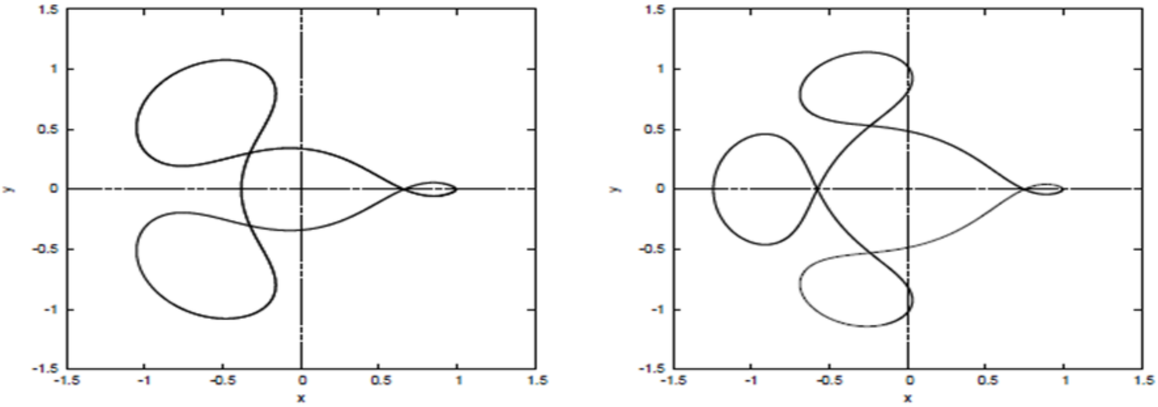

Figure 3: Arenstorf periodic orbits for α = 0.012277471 and initial conditions

x(0) = 0.994, y(0) = 0, x˙(0) = 0;

left subfigure: y˙(0) = –2.03173262955733683573020579 24,

right subfigure: y˙(0) = –2.00158510637908252240537862224,

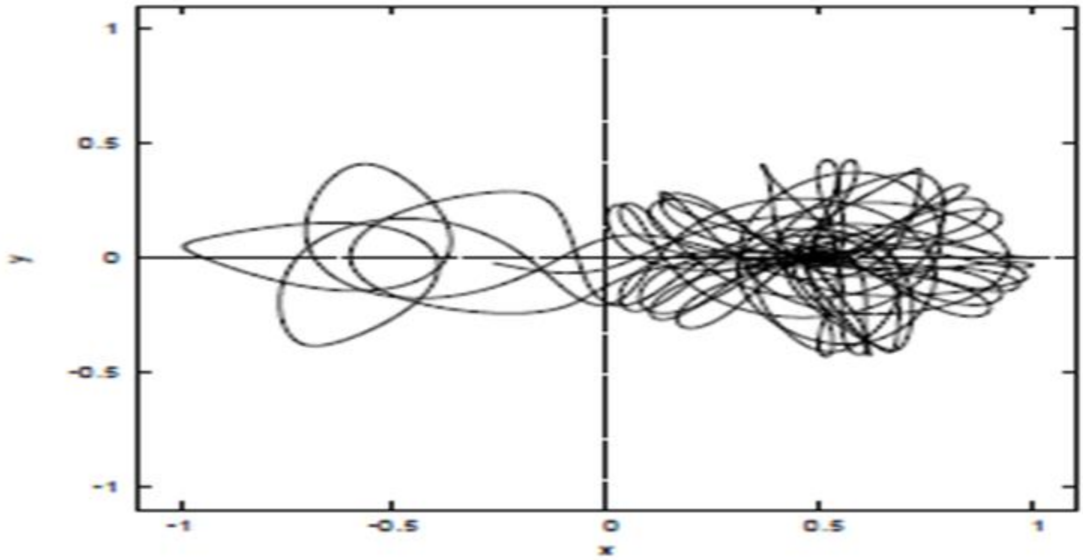

Figure 4: Chaotic orbit: α = 0.5, x(0) = 1, y(0) = 0, x˙(0) = 0, y˙(0) = 0.

Many studies were dedicated to the classical gravitational three-body problem, involving different methods and theories. The development of modern computers and computational techniques gave the possibility to deal with these problems using more powerful methods. This approach led to new results. Szebehely (1967), Marchal (1990) and many other researchers have dedicated extensive studies to this problem, pointing out various and interesting aspects.

V. CONCLUSION

In the study of the PCR3BP the Jacobian integral plays an important role, since it makes possible certain general, qualitative statements regarding the motion without actually solving the equations of motion. It permits for example the establishment of certain forbidden regions from which the third body is excluded. The application of this principle to celestial mechanics was first made by Hill (1878) showing that the Moon cannot depart from the Earth's neighbourhood arbitrarily far. These regions are called today Hill-regions.

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