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On A Special Type of Operator, Called λ-Jection of Third Order

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ABSTRACT

In this paper, an operator called λ-jection or λ-jection of third order has been considered. I investigate forms of an operator E in R^2 which satisfy conditions of λ-jection.

Keywords : λ-Jection, Projection, Trijection.

I. INTRODUCTION

Dr. P Chandra defined a trijection operator in his Ph.D. thesis titled "Investigation into the theory of operators and linear spaces". [1] In Dunford N. and Schwartz J. [2], p.37 and Rudin [3], p.126 a projection operator E has been defined as $E^2 = E$. In analogue to this, E has been defined a trijection operator if $E^3 = E$, which is a generalisation of projection operator.

I. Definition

Let L be a linear space and E a linear operator on L. We define E to be a λ-jection of third order or simply a λ-jection if

$$E^3 + \lambda E^2 = (1 + \lambda)E, \lambda \; being \; a \; scalar$$

In case λ=0, we have $E^3 = E$ i.e. E is a trijection. In case of a projection i.e. E is a trijection. In case of a projection i.e. $E^2 = E$, this condition is also satisfied i.e.- it is a **λ**-jection too.

II. Main Results

We investigate when an operator E on R^2 happens to be a **λ**-jection.

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Theorem 1

Let E be an operator defined on R^2 by

$$E(x,y) = (ax+by, cx+dy) \text{ with a,b,c,d in R.}$$

We find out conditions when E is a **λ**-jection.

Proof :-

We have

$$E^2(x,y) = E(E(x,y)) = E(ax+by, cx+dy)$$
$$= (a(ax+by)+b(cx+dy), c(ax+by)+d(cx+dy))$$
$$= ((a^2+bc)x+b(a+d)y, c(a+d)x+(bc+d^2)y)$$
$$= (Ax+By, Cx+Dy), say$$

$where\ A = a^2+bc, B = b(a+d), C = c(a+d), D = bc+d^2$

$Let\ ad-bc = m\ and\ a+d = n$

$Then\ d = n-a, Hence\ a(n-a)-bc = m$

$an-a^2-bc = m$

$So, A = a^2+bc = an-m\ \ and\ \ bc = an-m-a^2$

$Now\ B = b(a+d) = bn, C = c(a+d) = cn$

$D = bc+d^2 = an-m-a^2+(n-a)^2$

$$= an-m-a^2+n^2+a^2-2an = n^2-an-m$$

Now

$E^3(x,y) = E(E^2(x,y)) = E(Ax+By, Cx+Dy)$

$= (a(Ax+By)+b(Cx+Dy), c(Ax+by)+d(Cx+Dy))$

$= ((aA+bC)x+(aB+bD)y, (cA+dC)x+(cB+dD)y)$

$= (A_1x+B_1y, C_1x+D_1y), say$

Then $A_1 = aA+bC = a(an-m)+bcn = a^2n-am+n(an-a^2-m)$

$= an^2-mn-am$

$B_1 = aB+bD = abn+b(n^2-an-m) = b(n^2-m)$

$C_1 = cA+dC = c(an-m)+(n-a)cn = c(n^2-m)$

$D_1 = cB+dD = cbn+(n-a)(n^2-an-m)$

$= (an-m-a^2)n+n^3-an^2-mn-an^2+a^2n+am$

$= n^3-an^2-2mn+am$

We substitute these values in

$$E^3(x,y)+\lambda E^2(x,y) = (1+\lambda)E(x,y)$$

We get

$$(A_1x+B_1y, C_1x+D_1y)+\lambda(Ax+By, Cx+Dy) = (1+\lambda)(ax+by, cx+dy)$$

Equating co-efficients of x,y in both coordinates

$$A_1+\lambda A = a(1+\lambda), B_1+\lambda B = b(1+\lambda)$$

$$C_1 + \lambda C = c(1+\lambda), D_1 + \lambda D = d(1+\lambda)$$

Now $A_1 + \lambda A = a(1+\lambda)$

$$\Rightarrow an^2 - mn - am + \lambda(an - m) = a + a\lambda$$

$\Rightarrow an^2 - mn - am + \lambda an - \lambda m - a - a\lambda = 0$ ——---------- (1)

$$B_1 + \lambda B = b(1+\lambda)$$

$$\Rightarrow b(n^2 - m) + \lambda bn = b(1+\lambda)$$

$$\Rightarrow n^2 - m + \lambda n = 1 + \lambda, assuming\ b \neq 0$$

$\Rightarrow n^2 - m = 1 + \lambda - \lambda n$ ——---------- (2)

$$C_1 + \lambda C = c(1+\lambda)$$

$$\Rightarrow c(n^2 - m) + \lambda cn = c(1+\lambda)$$

$$\Rightarrow n^2 - m + \lambda n = 1 + \lambda, assuming\ c \neq 0$$

$$\Rightarrow n^2 - m = 1 + \lambda - \lambda n, which\ is\ same\ as\ (2)$$

$$D_1 + \lambda D = d(1+\lambda)$$

$$\Rightarrow n^3 - an^2 - 2mn + am + \lambda(n^2 - an - m) = (1+\lambda)(n-a)$$

$$\Rightarrow n^3 - an^2 - 2mn + am + \lambda n^2 - a\lambda n - \lambda m) = n(1+\lambda) - a - a\lambda$$

$\Rightarrow n^3 - a(n^2 - m) - 2mn + \lambda(n^2 - m) - a\lambda n = n + n\lambda - a - a\lambda$ ——---------- (3)

(1) Can be put in form,

$$a(n^2 - m) - mn + \lambda an - \lambda m - a - a\lambda = 0$$

Using (2), we get

$$a(1 + \lambda - \lambda n) - mn + \lambda an - \lambda m - a - a\lambda = 0$$

$$\Rightarrow -mn - \lambda m = 0$$

$$\Rightarrow mn + \lambda m = 0$$

$$\Rightarrow m(n+\lambda) = 0$$

$\Rightarrow m = 0\ or\ n = -\boldsymbol{\lambda}$ ——---------- (4)

Using (2) in relation (3), we get

$$n^3 - a(1 + \lambda - \lambda n) - 2mn + \lambda(1 + \lambda - \lambda n) - a\lambda n = n + \lambda n - a - a\lambda$$

$$n^3 - a - a\lambda + a\lambda n - 2mn + \lambda + \lambda^2 - \lambda^2 n - a\lambda n = n + \lambda n - a - a\lambda$$

$n^3 - 2mn + \lambda + \lambda^2 - \lambda^2 n - n - \lambda n = 0$ ——---------- (5)

From (4), let m=0 and put in (5), then we get

$$n^3 + \lambda + \lambda^2 - \lambda^2 n - n - \lambda n = 0$$

$$\Rightarrow n^3 - n(1 + \lambda + \lambda^2) + \lambda + \lambda^2 = 0$$

This is a cubic in n and roots are easily found to be 1, $\boldsymbol{\lambda}$ and -(1+$\boldsymbol{\lambda}$)

Hence when m=0, n=1, $\boldsymbol{\lambda}$, –(1+$\boldsymbol{\lambda}$)

From (4), let n = –$\boldsymbol{\lambda}$, then due to (5),

$$-\lambda^3 + 2m\lambda + \lambda + \lambda^2 + \lambda^3 + \lambda + \lambda^2 = 0$$

$$\Rightarrow 2m\lambda + 2\lambda + 2\lambda^2 = 0$$

$$\Rightarrow m\lambda + \lambda + \lambda^2 = 0$$

$$\Rightarrow \lambda(m + 1 + \lambda) = 0$$

$$\lambda = 0 \; or \; \lambda = -(m + 1), i.e. m = -(1 + \lambda)$$

If we choose $\lambda = 0 \; then \; n = 0$ (Due to 4)

Due to (2), -m = 1 or m = –1

So, we have the case m = –1, n=0

Otherwise, we get the case when m= –(1+$\boldsymbol{\lambda}$) and n = –$\boldsymbol{\lambda}$

Thus finally we see that (m,n) takes values (0,1), (0,$\boldsymbol{\lambda}$), (0, –(1+$\boldsymbol{\lambda}$)), (–1,0) and (–(1+$\boldsymbol{\lambda}$), –$\boldsymbol{\lambda}$)

So, when E is a $\boldsymbol{\lambda}$-jection, we have the above possibilities.

Theorem 2

Let m=0, n=1, then

$$E(x, y) \; = \; (ax + by, cx + (1 - a)y) \; where \; bc = a - a^2$$

$$Also \; E^2 = E \; so \; E \; is \; a \; projection$$

Proof :-

Due to (2), $n^2 - m = 1 + \lambda - \lambda n$

$\Rightarrow 1 = 1 + \lambda - \lambda$ which is true for all values of λ

Also $m = 0 \Rightarrow ad = bc$

$$n = 1 \Rightarrow a + d = 1 \Rightarrow d = 1 - a$$

Hence $ad = bc \Rightarrow a(1 - a) = bc \Rightarrow a = a^2 + bc = A$

Also, $B = bn = b, C = cn = c$

$$D = n^2 - an - m = 1 - a = d$$

So, in this case,

$$E(x, y) = (ax + by, cx + (1 - a)y) \; where \; bc = a - a^2$$

Also, $E^2(x, y) = (Ax + By, Cx + Dy) = (ax + by, cx + dy) = E(x, y)$

Hence $E^2 = E$, So E is a projection.

Theorem 3

Let m=0 and n=$\boldsymbol{\lambda}$. Then $\boldsymbol{\lambda}$ has two values 1 and –1/2

When $\boldsymbol{\lambda}$ = 1, we have case as in theorem 2, when $\boldsymbol{\lambda}$ = –1/2

Then,

$$E(x, y) = (ax + by, cx + (-\frac{1}{2} - a)y)$$

$$where \; bc = -\frac{1}{2}a - a^2$$

Also in this case, $E^2 = -\frac{1}{2}E$

Proof :-

Let m=0, n=$\boldsymbol{\lambda}$

Due to (2), $\lambda^2 = 1 + \lambda - \lambda^2$

$$\Rightarrow 2\lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = 1 \; or \; -\frac{1}{2}$$

So, consider m=0, n=1

We have already considered this case in theorem 2 and E is a projection.

Now consider the case $m = 0, n = \lambda = -\frac{1}{2}$

Then $a + d = -\frac{1}{2} \Rightarrow d = -\frac{1}{2} - a \quad (\because \; n = a + d)$

So $ad = bc \Rightarrow a(-\frac{1}{2} - a) = bc \quad (\because \; m = ad - bc = 0)$

$$\Rightarrow A = a^2 + bc = -\frac{1}{2}a$$

$$B = bn = -\frac{1}{2}b, C = cn = -\frac{1}{2}c$$

$$D = n^2 - an - m = n^2 - an = n(n - a) = -\frac{1}{2}(-\frac{1}{2} - a) = -\frac{1}{2}d$$

Hence in this case,

$$E(x, y) = (ax + by, cx + (-\frac{1}{2} - a)y)$$

where $bc = -\frac{1}{2}a - a^2$

Also, $E^2(x, y) = (Ax + By, Cx + Dy) = (-\frac{1}{2}ax - \frac{1}{2}by, -\frac{1}{2}cx - \frac{1}{2}dy)$

$$= -\frac{1}{2}(ax + by, cx + dy) = -\frac{1}{2}E(x, y)$$

Hence, in this case, $E^2 = -\frac{1}{2}E$

Theorem 4

Let $m = 0, n = -(\lambda + 1)$ In this case,

$$E(x, y) = (ax + by, cx - (a + 1 + \lambda)y)$$

$$where \; bc = -a(1 + \lambda) - a^2$$

Also, $E^2 = -(\lambda + 1)E$

Proof:-

Due to (2),

$$(\lambda + 1)^2 = 1 + \lambda + \lambda(1 + \lambda) = 1 + 2\lambda + \lambda^2$$

Which is true for all values of λ. Here $a + d = -(\lambda + 1)$

Now $bc = an - m - a^2 = -a(\lambda + 1) - a^2$

$$A = a^2 + bc = -a(\lambda + 1)$$

$$B = bn = -b(\lambda + 1)$$

$$C = cn = -c(\lambda + 1)$$

$$D = n^2 - an - m = n(n - a) = nd = -d(\lambda + 1)$$

So in this case,

$$E(x,y) = (ax + by, cx - (a + 1 + \lambda)y)$$

where $bc = -a(1 + \lambda) - a^2$

$$E^2(x,y) = (Ax + By, Cx + Dy)$$
$$= (-a(\lambda + 1)x - b(\lambda + 1)y, -c(\lambda + 1)x - d(\lambda + 1)y)$$
$$= -(\lambda + 1)(ax + by, cx + dy)$$
$$= -(\lambda + 1)E(x,y)$$

Hence in this case,

$$E^2 = -(\lambda + 1)E$$

Corollary

If m=0, n=0 then

$$E(x,y) = (ax + by, cx - ay)$$
$$where, bc = -a^2 \ and \ E^2 = 0$$

Proof :-

In above theorem put $\boldsymbol{\lambda} = -1$

Theorem 5

Let m = -1 and n=0. In this case

$$E(x,y) = (ax + by, cx - ay), where \ bc = 1 - a^2$$

Moreover, $E^2 = I$ Identity Operator

Proof:-

Since n=0, a+d=0. Hence d=-a

$$m = -1 \Rightarrow ad - bc = -1$$
$$\Rightarrow a(-a) - bc = -1$$
$$\Rightarrow A = a^2 + bc = 1$$
$$Also, B = bn = 0, C = cn = 0$$
$$D = bc + d^2 = bc + (-a)^2 = a^2 + bc = 1$$

Hence $E(x,y) = (ax + by, cx + dy)$

$$where \ bc = 1 - a^2$$

And $E^2(x,y) = (Ax + By, Cx + Dy) = (x,y) = I(x,y)$

Thus $E^2 = I$

Theorem 6

Let $m = -(1 + \lambda) \ and \ n = -\lambda$

Then $E(x,y) = (ax + by, cx - (a + \lambda)y)$

Where $bc = 1 + \lambda - a\lambda - a^2$

And $E^2 = -\lambda E + (1 + \lambda)I$

Proof:-

Due to (2),

$\lambda^2 + 1 + \lambda = 1 + \lambda - \lambda(-\lambda)$ which is true for any λ

Now $n = -\lambda \Rightarrow a + d = -\lambda$

$$\Rightarrow d = -\lambda - a$$

Also $m = ad - bc$

$$\Rightarrow -(1+\lambda) = a(-\lambda - a) - bc$$

$$\Rightarrow bc = (1+\lambda) - a\lambda - a^2$$

Hence $E(x, y) = (ax + by, cx + dy)$

where $bc = (1+\lambda) - a\lambda - a^2$

Also, $A = a^2 + bc = (1+\lambda) - a\lambda = an - m$

$$B = bn$$

$$C = cn$$

$$D = n^2 - an - m = n(n-a) - m = nd - m$$

So $E^2(x, y) = ((an - m)x + bny, cnx + (nd - m)y)$

$$= (anx + bny, cnx + dny) - (mx, my)$$

$$= n(ax + by, cx + dy) - m(x, y)$$

$$= nE(x, y) - mI(x, y)$$

Hence $E^2 = nE - mI = -\lambda E + (1+\lambda)I$

We see that above equation multiplied by E gives

$$E^3 = -\lambda E^2 + (1+\lambda)E$$

which is true

Theorem 7

In case b = c = 0,

$$E(x, y) = (ax, dy) \; with \; bc = 0$$

Also, a and d takes values in the set $\{0,1, -(\lambda + 1)\}$

This gives 9 possibilities for E(x,y)

Proof:-

Since b = c = 0, E(x,y) = (ax,dy)

Hence $E^2(x, y) = (a^2x, d^2y), E^3(x, y) = (a^3x, d^3y)$

Substituting in condition for $\boldsymbol{\lambda}$-jection

$$(a^3x, d^3y) + \lambda(a^2x, d^2y) = (1+\lambda)(ax, dy)$$

$$\Rightarrow (a^3x + \lambda a^2x, d^3y + \lambda d^2y) = ((1+\lambda)ax, (1+\lambda)dy)$$

Comparing coefficients of x,y

$$a^3 + \lambda a^2 = (1+\lambda)a$$

And $d^3 + \lambda d^2 = (1+\lambda)d$

Now $a^3 + \lambda a^2 - (1+\lambda)a = 0$

$$\Rightarrow a[a^2 + \lambda a - (1+\lambda)] = 0$$

$$\Rightarrow a(a-1)(a+1+\lambda) = 0$$

$$a = 0,1,-(\lambda+1)$$

Similarly, $d = 0,1,-(\lambda+1)$

Thus, a and d takes values in the set {0,1, -(λ+1)}

So we have nine possibilities for E(x,y). These are given by

$$E(x,y) = (0,0)\ where\ a = d = 0. Thus\ E\ is\ 0\ operator.$$

$$E(x,y) = (0,y)\ where\ a = 0, d = 1. This\ is\ a\ projection.$$

$$E(x,y) = (0,-(\lambda+1)y)\ when\ a = 0, d = -(\lambda+1)$$

$$Similarly\ E(x,y) = (x,0)\ which\ is\ a\ projection.$$

$$E(x,y) = (x,y)\ i.e. -E = I$$

$$E(x,y) = (x,-(\lambda+1)y)$$

$$E(x,y) = (-(\lambda+1)x,0)$$

$$E(x,y) = (-(\lambda+1)x,y)$$

$$E(x,y) = (-(\lambda+1)x,-(\lambda+1)y), i.e. -E = -(\lambda+1)I$$

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