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Spectral Resolution of a $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ - jection of Third Order

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ABSTRACT

In this paper I consider an operator E which is a ($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$)-jection of third order and obtain its spectral resolution.

Keywords:- ($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$)-jection, projection, spectrum, spectral resolution

I. INTRODUCTION

A trijection operator was introduced by Dr. P. Chandra in his Ph.D. thesis titled "Investigation into the theory of operators and linear spaces".[1] A projection operator E is defined as E^2=E in DUnford and Schwarz [2], p.37 and Rudin [3], p.126. E is a trijection operator if E^3=E. I have defined E to be a $\boldsymbol{\lambda}$-jection of third order [5], if

$E^3 + \lambda E^2 = (1 + \lambda)E, \lambda\ being\ a\ scalar$

To generalise it further, I have defined E to be a ($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$)-jection if

$\lambda E^3 + \mu E^2 = (\lambda + \mu)E$

$\boldsymbol{\lambda}$,$\boldsymbol{\mu}$ being scalars.

I. Definition

Let H be a Hilbert space and E an operator on H. Let $\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_m$ be eigen values of E and $M_1, M_2, M_3, \ldots\ldots. M_m$ be their corresponding eigen spaces. Let $P_1, P_2, P_3, \ldots\ldots., P_m$ be the projections on these eigen spaces. Then according to definition of spectral theorem in SImmons [4], pp 279-290, the following statements are all equivalent to one another.

1. The M_i^s are pairwise orthogonal and span H.
2. The P_i^s are pairwise orthogonal, $I = \sum_{i=1}^{m} \quad P_i\ and\ E = \sum_{i=1}^{m} \quad \lambda_i P_i$
3. E is normal.

   Then the set of eigen values of E is called its spectrum and is denoted by $\boldsymbol{\sigma}$(E). Also

   $E = \lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots. + \lambda_m P_m$

Expression for E given above is called the spectral resolution of E.

II. Main Result

Theorem 1

Let E be a ($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$)-jection. Then there are two mutually orthogonal projections L and M such that $\boldsymbol{E = L - \frac{(\lambda+\mu)}{\lambda} M}$ where $\boldsymbol{L = \frac{\lambda E^2+(\lambda+\mu)E}{2\lambda+\mu}}$ and $\boldsymbol{M = \frac{\lambda^2(E^2-E)}{(\lambda+\mu)(2\lambda+\mu)}}$

assuming $\boldsymbol{\lambda \neq 0, \lambda+\mu \neq 0, 2\lambda+\mu \neq 0}$

Proof:-

Let us assume that E can be expressed in form aL+bM, L and M being two mutually orthogonal projections. This means

$E = aL + bM$

Where L^2=L, M^2=M, LM=0

a,b being scalars.

Hence

$aE = a^2L + abm$ -- (1)

and $E^2 = a^2L + b^2M$ -- (2)

From (1) and (2),

$aE - E^2 = abM - b^2M = b(a-b)M$

$$\Rightarrow M = \frac{aE - E^2}{b(a-b)} \text{ assuming } a \neq 0, b \neq 0, a - b \neq 0$$

So, $bM = \frac{aE-E^2}{a-b}$

Hence $aL = E - bM = E - \frac{(aE-E^2)}{a-b} = \frac{E^2-bE}{a-b}$

$$\Rightarrow L = \frac{E^2 - bE}{a(a-b)}$$

Since LM = 0, we have

$$\frac{E^2 - bE}{a(a-b)} * \frac{aE - E^2}{b(a-b)} = 0$$

$\Rightarrow (E^2 - bE)(aE - E^2) = 0$

$\Rightarrow aE^3 - E^4 - abE^2 + bE^3 = 0$

$\Rightarrow E^4 = (a+b)E^3 - abE^2$ -- (3)

But E being a ($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$)-jection

$\lambda E^3 = (\lambda+\mu)E - \mu E^2$

$\Rightarrow \lambda E^4 = (\lambda+\mu)E^2 - \mu E^3$

$\Rightarrow E^4 = \frac{\lambda+\mu}{\lambda}E^2 - \frac{\mu}{\lambda}E^3 \ (\lambda \neq 0)$ -- (4)

Comparing (3) and (4),

$$a + b = \frac{-\mu}{\lambda} \text{ and } ab = \frac{-(\lambda+\mu)}{\lambda}$$

Solving for a and b,

$$a = 1 \text{ and } b = \frac{-(\lambda+\mu)}{\lambda}$$

Hence $E = L - \frac{(\lambda+\mu)}{\lambda}M$

Also $L = \frac{E^2 - bE}{a(a-b)} = \frac{E^2 + \frac{(\lambda+\mu)}{\lambda}E}{1+\frac{\lambda+\mu}{\lambda}} = \frac{\lambda E^2 + (\lambda+\mu)E}{2\lambda+\mu}$ $assuming\ 2\lambda + \mu \neq 0$

$$M = \frac{aE - E^2}{b(a-b)} = \frac{E - E^2}{\frac{-(\lambda+\mu)}{\lambda}(1 + \frac{\lambda+\mu}{\lambda})} = \frac{\lambda^2(E^2 - E)}{(\lambda+\mu)(2\lambda+\mu)}\ assuming\ \lambda + \mu \neq 0$$

Note:- Since L, M are projections such that LM=0 their sum L+M is also a projection. Call it N, then

$$N = L + M = \frac{\lambda E^2 + (\lambda+\mu)E}{2\lambda+\mu} + \frac{\lambda^2(E^2 - E)}{(\lambda+\mu)(2\lambda+\mu)} = \frac{\lambda E^2 + \mu E}{\lambda+\mu}$$

Theorem 2

Let E be a ($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$)-jection on a Hilbert space H. Then there are three pairwise orthogonal projections $\boldsymbol{P_1, P_2, P_3}$ *such that*

$$\boldsymbol{E = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3}$$

where $\boldsymbol{\lambda_1, \lambda_2, \lambda_3}$ are scalars and $\boldsymbol{I = P_1 + P_2 + P_3}$

Proof:-

Due to theorem I

$$E = L - \frac{(\lambda+\mu)}{\lambda}M$$

Call L as P_1, M as P_2 and I-N as P_3

Then $P_1 + P_2 + P_3 = L + M + I - N = I$

Also $P_1 P_2 = LM = 0$

$P_1 P_3 = L(I - L - M) = L - L^2 = 0$

$P_2 P_3 = M(I - L - M) = M - M^2 = 0$

So, P_1, P_2, P_3 are pairwise orthogonal projections.

Choose $\lambda_1 = 1, \lambda_2 = \frac{-(\lambda+\mu)}{\lambda}$ $and\ \lambda_3 = 0$

Then $E = P_1 - \frac{(\lambda+\mu)}{\lambda}P_2 = P_1 - \frac{(\lambda+\mu)}{\lambda}P_2 + 0.P_3$

Choose $\lambda_1 = 1, \lambda_2 = \frac{-(\lambda+\mu)}{\lambda}$ $and\ \lambda_3 = 0$

Then $E = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$

Theorem 3

Range of projection P_1 denoted by $\boldsymbol{R_{P_1}}$ is given by $\boldsymbol{R_{P_1} = \{z: Ez = z\} = M_1(say)}$

Proof:-

Let z be an element of R_{P_1}, then since P_1 is a projection, $P_1 z=z$

Now $EP_1 = E\frac{(\lambda E^2 + (\lambda+\mu)E)}{2\lambda+\mu} = \frac{\lambda E^3 + (\lambda+\mu)E^2}{2\lambda+\mu} = \frac{\lambda E^3 + \mu E^2 + \lambda E^2}{2\lambda+\mu} = \frac{(\lambda+\mu)E + \lambda E^2}{2\lambda+\mu} = P_1$

Hence $Ez = E(P_1 z) = EP_1 z = P_1 z = z$

So $z \in M_1$

Hence $R_{P_1} \subseteq M_1$ ——--- (5)

Conversely, let $z \in M_1$, then Ez=z

Then $E^2 z = E(Ez) = Ez = z$

So, $P_1 z = \frac{(\lambda E^2+(\lambda+\mu)E)}{2\lambda+\mu} z = \frac{\lambda z+(\lambda+\mu)z}{2\lambda+\mu} = \frac{2\lambda+\mu}{2\lambda+\mu} z = z$

Hence $z \in R_{P_1}$

Thus $M_1 \subseteq R_{P_1}$ -- (6)

From (5) and (6)

$R_{P_1} = M_1$

Theorem 4

We show that

$\boldsymbol{R_{P_2} = \{z{:}\, Ez = -(\frac{\mu}{\lambda} + 1)z\} = M_2\ (say)}$

Proof:-

Let $z \in R_{P_2}, then\ P_2 z = z$

Also $EP_2 = \frac{E(\lambda^2(E^2-E))}{(\lambda+\mu)(2\lambda+\mu)} = \frac{\lambda^2(E^3-E^2)}{(\lambda+\mu)(2\lambda+\mu)}$

but $\lambda E^3 - \lambda E^2 = -\mu E^2 - \lambda E^2 + (\lambda+\mu)E = (\lambda+\mu)(E - E^2)$

Hence $EP_2 = \frac{\lambda(\lambda+\mu)(E-E^2)}{(\lambda+\mu)(2\lambda+\mu)} = \frac{\lambda(E-E^2)}{2\lambda+\mu} = \frac{-\lambda^2(E^2-E)}{(\lambda+\mu)(2\lambda+\mu)} * \frac{(\lambda+\mu)}{\lambda} = -(1+\frac{\mu}{\lambda})P_2$

So $Ez = EP_2 z = -(1+\frac{\mu}{\lambda})P_2 z = -(1+\frac{\mu}{\lambda})z$

Thus $z \in M_2$

Hence $R_{P_2} \subseteq M_2$ -- (7)

Conversely, let $z \in M_2$, then

$Ez = -(\frac{\mu}{\lambda} + 1)z$

$E^2 z = E(Ez) = -(\frac{\mu}{\lambda} + 1)Ez = (\frac{\mu}{\lambda} + 1)^2 z$

So $E^2 z - Ez = [(\frac{\mu}{\lambda}+1)^2 + (\frac{\mu}{\lambda}+1)]z = (\frac{\mu}{\lambda}+1)(\frac{\mu}{\lambda}+2)z$

$= \frac{(\lambda+\mu)(2\lambda+\mu)}{\lambda^2} z$

Hence $P_2 z = \frac{\lambda^2(E^2-E)z}{(\lambda+\mu)(2\lambda+\mu)} = \frac{(\lambda+\mu)(2\lambda+\mu)}{(\lambda+\mu)(2\lambda+\mu)} z = z$

i.e. $z \in R_{P_2}$

Hence $M_2 \subseteq R_{P_2}$ -- (8)

Therefore from (7) and (8),

$R_{P_2} = M_2$

Theorem 5

We show that

$\boldsymbol{R_{P_3} = \{z{:}\, Ez = 0\} = M_3(say)}$

Proof:-

We have

$P_3 = I - N = I - P_1 - P_2$

Let $z \in R_{P_3}$, then $P_3 z = z$

Now $EP_3 = E(I - P_1 - P_2) = E - EP_1 - EP_2$
$= E - P_1 + (1 + \frac{\mu}{\lambda})P_2 = E - (P_1 - (1 + \frac{\mu}{\lambda})P_2)$
$= E - E = 0$
So $Ez = E(P_3 z) = (EP_3)z = 0z = 0$
Thus $z \in M_3$
Hence $R_{P_3} \subseteq M_3$ -- (9)
Conversely, let $z \in M_3$ then $Ez = 0$
$\Rightarrow E^2 z = 0 \Rightarrow E^3 z = 0$
Hence $P_3 z = [I - \frac{(\lambda E^2 + \mu E)}{\lambda + \mu}]z = z - 0 = z$
i.e. $z \in R_{P_3}$
So, $M_3 \subseteq R_{P_3}$ --- (10)
From (9) and (10),
$R_{P_3} = M_3$

Theorem 6

Let E be a ($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$)-jection on a Hilbert space H.
Let $\lambda_1 = 1, \lambda_2 = -(1 + \frac{\mu}{\lambda})\ and\ \lambda_3 = 0$ be eigen values of E
Let $M_1, M_2, M_3\ be\ their\ corresponding\ eigen\ spaces.$
Let P_1, P_2, P_3 be projections on these eigen spaces where
$$P_1 = \frac{\lambda E^2 + (\lambda + \mu)E}{2\lambda + \mu}, P_2 = \frac{\lambda^2(E^2 - E)}{(\lambda + \mu)(2\lambda + \mu)}, P_3 = I - (\frac{\lambda E^2 + \mu E}{\lambda + \mu})$$
Then $P_1 + P_2 + P_3 = I$
P_i^s are pairwise orthogonal and spectral resolution of E is given by (assuming $\lambda \neq 0, \lambda + \mu \neq 0, 2\lambda + \mu \neq 0$)
$E = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$
Spectrum of E is given by $\{1, -(\frac{\mu}{\lambda} + 1), 0\}$

Proof:-
Theorem 3,4,5 show that $\lambda_1 = 1, \lambda_2 = -(1 + \frac{\mu}{\lambda})\ and\ \lambda_3 = 0$ are eigen values of E, M_1, M_2, M_3 are their corresponding eigen spaces and P_1, P_2, P_3 are pairwise orthogonal projections on these eigen spaces. Also due to theorem 2,
$E = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$
$and\ I = P_1 + P_2 + P_3$
Hence expression for E given above is the spectral resolution of E. Since the eigen values of E are $1, -(1 + \frac{\mu}{\lambda})\ and\ 0$, spectrum of E is given by
$$\sigma(E) = \{1, -(1 + \frac{\mu}{\lambda}), 0\}$$

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