International Journal of Scientific Research in Science and Technology
Print ISSN: 2395-6011 | Online ISSN: 2395-602X (www.ijsrst.com)
doi : https://doi.org/10.32628/IJSRST

Spectral Resolution of Square of a λ-jection of Fourth Order

Dr. Rajiv Kumar Mishra
Associate Professor
Department of Mathematics, Rajendra College Chapra
(J.P. University, Chapra, Bihar, PIN - 841301)
E-mail - dr.rkm65@gmail.com

Article Info
Volume 8, Issue 4
Page Number : 723-730
Publication Issue
July-August-2021
Article History
Received : 06 May 2021
Accepted : 20 July 2021
Published : 13 August 2021

ABSTRACT
In this paper I define a λ - jection of fourth order and obtain spectral resolution of square of such an operator.

Keywords – λ-jection of third order, λ-jection of fourth order, projection spectrum.

I. INTRODUCTION

Dr. P. Chandra introduced the concept of trijection in his Ph.D thesis titled "Investigation into the theory of operators and linear spaces".[1] An operator E is called a projection if $E^2 = E$ as given in Dunford and Schwarz [2],p.37 or Rudin [3], p.126. E is trijection operator if $E^3 = E$. I had defined E to be a λ-jection of third order [5] if

$E^3 + \lambda E^2 = (1+ \lambda)E$, λ being a scalar

To further extend this idea, I define E to be a λ - jection of fourth order if

$E^4 + \lambda E^3 = (1+ \lambda)E^2$, λ being a scalar

1. Definition

Let H be a Hilbert space and E an operator on H. Let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ be eigen values of E and $M_1, M_2, \ldots\ldots\ldots, M_m$ be their corresponding eigen spaces. Let $P_1, P_2, \ldots\ldots\ldots, P_m$ be the projection on these eigen spaces, Then according to definition of spectral theorem in Simmons [4],p. 279-290 , the following statements are all equivalent to one-another,

1) The M^i 's are pairwise orthogonal and span H.
2) The P^i 's are pairwise orthogonal, $I = \sum_{i=1}^{m} P_i$ and $E = \sum_{i=1}^{m} \lambda_i P_i$
3) E is normal

Then the set of eigen values of E is called its spectrum ,denoted by $\sigma(E)$. Also if $E= \lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots + \lambda_m P_m$

Then expression for E given above is called the spectral resolution of E.

II. MAIN RESULT

Theorem 1

Let E be a λ jection of 4^{th} order.Then E^2 can be expressed as a linear combination of three pairwise orthogonal projections.

(where $\lambda \neq 0,-1$ or-2)

Proof:

First, we examine when $aE^3 + bE^2$ is a projection, a,b being scalars, i.e.-

$(aE^3 + bE^2)^2 = aE^3 + bE^2$

$\Rightarrow a^2E^6 + b^2E^4 + 2abE^5 = aE^3 + bE^2$ --- (1)

So we need to find E^5 and E^6 in terms of E^2 and E^3.

We have

$E^4 + \lambda E^3 = (1 + \lambda)E^2$

Let $\mu = \lambda + 1, then\ \lambda = \mu - 1$

Then $E^4 + (\mu - 1)E^3 = \mu E^2$

Also since $\lambda \neq 0, -1\ or\ -2$

$\mu \neq 1{,}0\ or\ -1$

Now $E^4 - E^3 = \mu(E^2 - E^3)$

Thus $E^4 = \mu E^2 + (1 - \mu)E^3$ -- (2)

Applying E to both sides,

$E^5 = \mu E^3 + (1 - \mu)E^4 = \mu E^3 + (1 - \mu)[\mu E^2 + (1 - \mu)E^3]$

$= (\mu - \mu^2)E^2 + (\mu + (1 - \mu)^2)E^3$

$= (\mu - \mu^2)E^2 + (1 - \mu + \mu^2)E^3$ -- (3)

Applying E to both sides

$E^6 = (\mu - \mu^2)E^4 + (1 - \mu + \mu^2)E^4$

$= (\mu - \mu^2)E^3 + (1 - \mu + \mu^2)[\mu E^2 + (1 - \mu)E^3]$

$= (\mu - \mu^2 + \mu^3)E^2 + [(\mu - \mu^2) + (1 - \mu + \mu^2)(1 - \mu)]E^3$

$= (\mu - \mu^2 + \mu^3)E^2 + (1 - \mu + \mu^2 - \mu^3)E^3$ -- (4)

In relation (1), we put values of E^4, E^5, E^6 from equations (2), (3) and (4), and get

$a^2[(\mu - \mu^2 + \mu^3)E^2 + (1 - \mu + \mu^2 - \mu^3)E^3] + b^2[\mu E^2 + (1 - \mu)E^3]$

$+2ab[(\mu - \mu^2)E^2 + (1 - \mu + \mu^2)E^3] = aE^3 + bE^2$

Equating co-efficients of E^2 on both sides,

$a^2(\mu - \mu^2 + \mu^3) + b^2\mu + 2ab(\mu - \mu^2) = b$ -- (5)

Equating co-efficients of E^3 on both sides,

$a^2(1 - \mu + \mu^2 - \mu^3) + b^2(1 - \mu) + 2ab(1 - \mu + \mu^2) = a$ -------------------- (6)

Adding (5) and (6),

$a^2 + b^2 + 2ab = a + b$

$\Rightarrow (a + b)^2 = (a + b)$

$\Rightarrow a + b = 0 \ or \ 1$

$Let \ a + b = 0, then \ b = -a$

Then from (5),

$a^2(\mu - \mu^2 + \mu^3) + a^2\mu - 2a^2(\mu - \mu^2) = -a$

Let $a \neq 0$, then

$a[\mu - \mu^2 + \mu^3 + \mu - 2\mu + 2\mu^2] = -1$

$$\Rightarrow a(\mu^2 + \mu^3) = -1 \Rightarrow a = \frac{-1}{\mu^2 + \mu^3} = \frac{-1}{\mu^2(1 + \mu)} \qquad (\mu \neq 0, -1)$$

$b = -a = \frac{-1}{\mu^2+\mu^3}$

Hence $aE^3 + bE^2 = \frac{E^2-E^3}{\mu^2+\mu^3} = \frac{E^2-E^3}{\mu^2(1+\mu)}$

Next let $a + b = 1, Then \ b = 1 - a$

Due to (5),

$a^2(\mu - \mu^2 + \mu^3) + (1 - a^2)\mu + 2a(1 - a)(\mu - \mu^2) = 1 - a$

$\Rightarrow \mu\{a^2 + (1 - a)^2 + 2a(1 - a)\} - \mu^2(a^2 + 2a(1 - a) + a^2\mu^3) = 1 - a$

$\Rightarrow \mu\{a + (1 - a)\}^2 - \mu^2(2a - a^2) + a^2\mu^3 = 1 - a$

$\Rightarrow \mu - 2a\mu^2 + a^2\mu^2 + a^2\mu^3 = 1 - a$

$\Rightarrow a^2(\mu^2 + \mu^3) - (2\mu^2 - 1)a + \mu - 1 = 0$

Hence $a = \frac{(2\mu^2-1)\pm\sqrt{(2\mu^2-1)^2-4(\mu^2+\mu^3)(\mu-1)}}{2(\mu^2+\mu^3)} = \frac{2\mu^2-1\pm1}{2(\mu^2+\mu^3)}$

$$= \frac{2\mu^2}{2(\mu^2 + \mu^3)} \; or \; \frac{2\mu^2 - 2}{2(\mu^2 + \mu^3)}$$

i.e. $a = \frac{1}{1+\mu} \; or \; \frac{\mu-1}{\mu^2}$

When $a = \frac{1}{1+\mu} \; then \; b = \frac{\mu}{1+\mu}$

$$So, aE^3 + bE^2 = \frac{E^3}{1+\mu} + \frac{\mu E^2}{1+\mu} = \frac{E^3 + \mu E^2}{1+\mu}$$

$$When \; a = \frac{\mu - 1}{\mu^2} \; then \; b = \frac{\mu^2 - \mu + 1}{\mu^2}$$

Then $aE^3 + bE^2 = \frac{(\mu-1)E^3 + (\mu^2-\mu+1)E^2}{\mu^2}$

So we have 3 projections which we name as

$$P_1 = \frac{E^3 + \mu E^2}{1+\mu}, P_2 = \frac{E^2 - E^3}{\mu^2(\mu+1)}$$

and $Q_3 = \frac{(\mu-1)E^3 + (\mu^2-\mu+1)E^2}{\mu^2}$

We also mark that

$$P_1 + P_2 = \frac{E^3 + \mu E^2}{1+\mu} + \frac{E^2 - E^3}{\mu^2(\mu+1)} = \frac{\mu^2(E^3 + \mu E^2) + E^2 - E^3}{\mu^2(\mu+1)}$$

$$= \frac{(\mu^2-1)E^3 + (\mu^3+1)E^2}{\mu^2(\mu+1)} = \frac{(\mu-1)E^3 + (1-\mu+\mu^2)E^2}{\mu^2} = Q_3$$

Let $P_3 = I - Q_3$ which is also a projection.

Then we see that

$$P_1 P_2 = \frac{E^3 + \mu E^2}{1+\mu} * \frac{E^2 - E^3}{\mu^2(\mu+1)} = \frac{(E^3 + \mu E^2)(E^2 - E^3)}{\mu^2(\mu+1)}$$

Now, numerator = $E^3(E^2 - E^3) + \mu E^2(E^2 - E^3)$

$= E^5 - E^6 + \mu E^4 - \mu E^5 = (1-\mu)E^5 - E^6 + \mu E^4$

$= (1-\mu)[\mu E^3 + (1-\mu)E^4] - (\mu - \mu^2)E^3 - (1 - \mu + \mu^2)E^4 + \mu E^4$

$= (\mu - \mu^2)E^3 + (1-\mu)^2 E^4 - (\mu - \mu^2)E^3 - (1 - \mu + \mu^2)E^4 + \mu E^4$

$= E^4[(1-\mu)^2 - 1 + \mu - \mu^2 + \mu] = 0$

Thus $P_1 P_2 = 0, i.e. P_1 P_2 \; are \; orthogonal$

$P_1 P_3 = P_1[I - Q_3] = P_1(I - P_1 - P_2) = P_1 - {P_1}^2 = 0$

$P_2 P_3 = P_2[I - Q_3] = P_2(I - P_1 - P_2) = P_2 - {P_2}^2 = 0$

Thus $P_1, P_2, P_3 \; are \; pairwise \; orthogonal$

Also

$$P_1 + \mu^2 P_2 = \frac{E^3 + \mu E^2}{1 + \mu} + \frac{E^2 - E^3}{1 + \mu} = \frac{(\mu + 1)E^2}{1 + \mu} = E^2$$

Also $P_1 + P_2 + P_3 = P_1 + P_2 + I - Q_3 = P_1 + P_2 + I - (P_1 + P_2) = I$

So $E^2 = P_1 + \mu^2 P_2 = P_1 + \mu^2 P_2 + 0.P_3 = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$

where $\lambda_1 = 1, \lambda_2 = \mu^2, \lambda_3 = 0$

Thus E^2 is a linear combination of three pairwise orthogonal projections.

Theorem 2

Let R_{P_1} be the range of P_1. Then

$\boldsymbol{R_{P_1} = \{z: P_1 z = z\} = \{z: E^2 z = z\} = M_1 (say)}$

Proof:-

Let $z \in R_{P_1}$, then since P_1 is a projection, $P_1 z = z$

Now $E^2 P_1 = \frac{E^2(E^3 + \mu E^2)}{1+\mu} = \frac{E^5 + \mu E^4}{1+\mu} = \frac{E^4 + \mu E^3}{1+\mu}$

$$= \frac{\mu E^2 + (1 - \mu)E^3 + \mu E^3}{1 + \mu} = \frac{\mu E^2 + E^3}{1 + \mu} = P_1$$

Hence $E^2 z = E^2 P_1 z = P_1 z = z \;\; i.e. z \in M_1$

Therefore $R_{P_1} \subseteq M_1$ ——--- (7)

Conversely, let $z \in M_1, i.e. E^2 z = z$

Then $E^3 z = E(E^2 z) = Ez$

$\Rightarrow E^4 z = E^2 z = z$

Now $E^4 z = \mu E^2 z + (1 - \mu)E^3 z$

$\Rightarrow z = \mu z + (1 - \mu)E^3 z$

$\Rightarrow (1 - \mu)E^3 z = (1 - \mu)z$

$\Rightarrow E^3 z = z \;\; since \; (1 - \mu) \neq 0$

Hence $P_1 z = \frac{E^3 z + \mu E^2 z}{1+\mu} = \frac{z + \mu z}{1+\mu} = z$

or $z \in R_{P_1}$

Thus $M_1 \subseteq R_{P_1}$ ——-- (8)

Hence from (7) and (8),

$R_{P_1} = M_1$

Theorem 3

We show that

$\boldsymbol{R_{P_2} = \{z: E^2 z = \mu^2 z\} = M_2(say)}$

Proof:-

Let $z \in R_{P_2}$ $then\ P_2 z = z$

Also, $E^2 P_2 = \frac{E^2(E^2-E^3)}{\mu^2(\mu+1)} = \frac{E^4-E^5}{\mu^2(\mu+1)} = \frac{\mu E^2+(1-\mu)E^3-E^5}{\mu^2(\mu+1)}$

$$= \frac{\mu E^2 + (1-\mu)E^3 - \{(\mu-\mu^2)E^2 + (1-\mu+\mu^2)E^3\}}{\mu^2(\mu+1)}\ using\ (2)\ and\ (3)$$

$$= \frac{\mu^2 E^2 - \mu^2 E^3}{\mu^2(\mu+1)} = \frac{(E^2-E^3)\mu^2}{\mu^2(\mu+1)} = \mu^2 P_2$$

So $E^2 P_2 z = \mu^2 P_2 z$

$\Rightarrow E^2 z = \mu^2 z$

$Thus\ z \in M_2$

Hence $R_{P_2} \subseteq M_2$ ———————————————————————————— (9)

Let $z \in M_2$ then $E^2 z = \mu^2 z \Rightarrow E^4 z = E^2(\mu^2 z) = \mu^4 z$

Hence $E^4 z = \mu E^2 z + (1-\mu)E^3 z$

$\Rightarrow \mu^4 z = \mu.\mu^2 z + (1-\mu)E^3 z$

$\Rightarrow \mu^3(\mu-1)z = (1-\mu)E^3 z$

$\Rightarrow E^3 z = -\mu^3 z$

Hence $P_2 z = \frac{(E^2-E^3)z}{\mu^2(\mu+1)} = \frac{\mu^2 z+\mu^3 z}{\mu^3+\mu^2} = z$

Thus $z \in R_{P_2}$

So, $M_2 \subseteq R_{P_2}$ ———————————————————————————— (10)

Due to (9) and (10)

$R_{P_2} = M_2$

Theorem 4

We show that

$\boldsymbol{R_{P_3} = \{z: E^2 z = 0\} = M_3(say)}$

Proof:-

Now $E^2 P_3 = E^2(I - Q_3) = E^2(I - P_1 - P_2)$

$= E^2 - E^2 P_1 - E^2 P_2 = E^2 - P_1 - \mu^2 P_2$

$= E^2 - (P_1 + \mu^2 P_2) = E^2 - E^2 = 0$ (due to theorem 1)

Let $z \in R_{P_3}$ then $P_3 z = z$

Hence $E^2 z = E^2 P_3 z = (E^2 P_3) z = 0z = 0$

So $z \in M_3$.

Hence $R_{P_3} \subseteq M_3$ …………… (11)

Conversely, let $z \in M_3$, then $E^2 z = 0 \Rightarrow E^3 z = 0$.

Hence $P_3 z = [I - P_1 - P_2] z$

Now, $P_1 z = \frac{(E^3 + \mu E^2)}{(\mu+1)} z = \frac{0}{\mu+1} = 0$

$P_2 z = \frac{(E^2 - E^3) z}{\mu^2(\mu+1)} = \frac{0}{\mu^2(\mu+1)} = 0$

So, $P_3 z = z - P_1 z - P_2 z = z$

Thus $z \in R_{P_3}$

Hence, $M_3 \subseteq R_{P_3}$……………… (12)

Due to (11) and (12),

$R_{P_3} = M_3$

Theorem 5

Let E be a λ-jection of fourth order (when $\lambda \neq 0$, -1 or -2) on a Hilbert space H. Then spectral resolution of E^2 is given by $E^2 = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$ where $\lambda_1 = 1$, $\lambda_2 = \mu^2$ and $\lambda_3 = 0$. Also P_1, P_2, P_3 are pairwise orthogonal projections such that $P_1 + P_2 + P_3 = I$. Also spectrum of E^2 is given by

$\sigma(E^2) = \{1, \mu^2, 0\}$.

Proof:-

Due to theorem 1,

$E^2 = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$

Where $\lambda_1 = 1$, $\lambda_2 = \mu^2$ and $\lambda_3 = 0$.

P_1, P_2, P_3 are pairwise orthogonal projections

Such that $P_1 + P_2 + P_3 = I$.

Due to theorems (2), (3) and (4), λ_1, λ_2 and λ_3 are eigen values of E^2 and M_1, M_2, M_3 are their corresponding eigen spaces. Hence,

$E^2 = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$

Gives spectral resolution of E^2.

Since eigen values of E^2 are 1, μ^2 and 0,

$\sigma(E^2) = \{1, \mu^2, 0\}$

III. REFERENCES

[1] Chandra, P.

"Investigation into the theory of operators and linear spaces" Ph.D. thesis, Patna University, 1977.

[2] Dunford, N. and Schwartz, J.

"Linear Operators, Part I", Interscience Publishers, Inc; New York, 1967, p. 37

[3] Rudin, W,

"Functional Analysis", Mc Graw Hill Book Company, Inc; New York, 1973, p.126.

[4] Simmons, G.F.:

"Introduction to Topology and Modern Analysis", McGraw Hill Book COmpany, Inc., New York, 1963

[5] Mishra, R.K.:

"On a special type of operators called **λ**-jection of third order", International Journal of Scientific Research in Science and Technology, (IJSRST), Online ISSN: 2395-602X, Print ISSN: 2395-6011, Volume 4, Issue 2, pp 2321-2328, January-February 2018

Cite this Article

Dr. Rajiv Kumar Mishra, "Spectral Resolution of Square of a λ-jection of Fourth Order", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 8 Issue 4, pp. 723-730, July-August 2021.

Journal URL : https://ijsrst.com/IJSRST1218433