On Infinitesimal Transformations in A Finsler Space

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A R T I C L E I N F O

A B S T R A C T

In the present communication studies have been carried out with special reference to infinitesimal projective and special projective transformations in a Finsler space and accordingly results have been derived in the form of theorems in a projective symmetric and non-symmetric Finsler space.

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1. INTRODUCTION

Berwald introduced a connection coefficient $G^i_{jk}(x,\dot{x})$ defined by

$$G^i_{jk}(x,\dot{x}) \overset{def.}{=} \frac{\partial^2 G^i}{\partial x^j \partial x^k} \qquad (1.1)$$

and accordingly the covariant derivative of an arbitrary covariant vector X^i in the sense of Berwald is given by Rund [4]

$$X^i_{(j)} = \frac{\partial X^i}{\partial x^j} - \frac{\partial X^i}{\partial \dot{x}^h} \frac{\partial G^h}{\partial \dot{x}^j} + G^i_{jh} X^h \quad . \qquad (1.2)$$

The functions G^i appearing in (1.2) are positively homogeneous of degree two in its directional arguments $\dot{x}^i$ and satisfies the following identities

$$\text{(a)} \quad G^i_{hkr}\dot{x}^r = G^i_{hkr}\dot{x}^k = G^i_{hkr}\dot{x}^h = 0 \,, \qquad (1.3)$$

$$\text{(b)} \ G^i_{hk}\dot{x}^h = 0 \ \text{ (c) } \ G^i_k\dot{x}^k = 2G^i \,.$$

The geodesic deviation has been defined in the following form

$$\frac{\partial^2 Z^j}{\partial u^2} + H_k^j(x, \dot{x}) x^k = 0,$$ (1.4)

where the vector Z^i is called the variation vector and the tensor $H_k^j(x, \dot{x})$ is being defined by

$$H_k^i = 2\partial_k G^i - \partial_h \dot{\partial}_k G^i \dot{x}^h + 2G_{kl}^i G^l - \dot{\partial}_l G^i \dot{\partial}_k G^l .$$ (1.5)

The tensors defined by

(a) $H_{jk}^i(x, \dot{x}) = \frac{1}{3}(\frac{\partial H_k^i}{\partial \dot{x}^j} - \frac{\partial H_j^i}{\partial \dot{x}^k})$ (1.6)

and (b) $H_{jkl}^i = \frac{\partial H_{jl}^i}{\partial \dot{x}^k}$

are respectively termed as Berwald's deviation tensor and Berwald's curvature tensor and they satisfy the following :

(a) $H_{khj}^k = H_{jh} - H_{hj}$,(b) $H_i \dot{x}^i = (n-1)H$ (1.7)

(c) $H_{ki}^j \dot{x}^k = H_i^j = H_{ik}^j \dot{x}^k$.

The projective covariant derivative of an arbitrary tensor $T_j^i(x, \dot{x})$ is given by Misra [2] as

$$T_{j((k))}^i = \partial_k T_j^i - \dot{\partial}_s T_j^s \prod_{rk}^i \dot{x}^r + T_j^h \prod_{hk}^i - T_h^i \prod_{jk}^h$$ (1.8)

where $\prod_{jk}^i(x, \dot{x}) \overset{def}{=} \{G_{jk}^i - \frac{1}{n+1}(2\delta_{(j}^i G_{<r>k)}^r + \dot{x}^i G_{rkg}^i\}$ (1.9)

are called projective connection coefficient and these coefficients are symmetric in its lower indices .
Involving the projective covariant derivative ,we have the following commutation formulae :

(a) $\partial_h(T_{j((k))}^i) - (\partial_h T_j^i)_{((k))} = T_j^s \prod_{shk}^i - T_s^i \prod_{jhk}^s$ (1.10)

and (b) $2T_{j[((h))((k))]}^i = -\dot{\partial}_r T_j^i Q_{shk}^r \dot{x}^s + T_j^s Q_{shk}^i - T_s^i Q_{jhk}^s$

where ,

$$Q_{hjk}^i \overset{def}{=} 2\{\partial_{[k}\prod_{j]h}^i - \prod_{rh[j}^i \prod_{k]}^r + \prod_{h[j}^r \prod_{k]r}^i\}$$ (1.11)

is called the projective entity and satisfies the following relations :

(a) $Q_{hjk}^i + Q_{jkh}^i + Q_{khj}^i = 0$, (1.12)

(b) $Q_{hjk((s))}^i + Q_{hks((j))}^i + Q_{hsj((k))}^i = 0$,

(c) $Q_{ijk}^i = Q_{jk}$ (d) $Q_{jk}^i = \frac{2}{3}\dot{\partial}_{[j} Q_{k]}^i$,

(e) $Q_{hjk}^i = \dot{\partial}_h Q_{jk}^i$, (f) $Q_{ijk}^i = Q_{jk}^i$, (g) $Q_k^i \dot{x}_k = 0$,

(h) $Q_{jk}^i = -Q_{kj}^i$ and (i) $Q_{hk}^i \dot{x}^h = Q_k^i$.

The projective connection coefficient $\prod_{jk}^i(x, \dot{x})$ satisfies the following relations :

(a) $\prod_{hkr}^i = \dot{\partial}_h \prod_{kr}^i$,(b)

$$\prod_{hk}^i = \dot{\partial}_h \prod_k^i,$$ (1.13)

(c) $\prod_{hkr}^i \dot{x}^h = 0$ and (d) $\prod_{hk}^i \dot{x}^h = \prod_k^i$

2. NON –AFFINE INFINITESIMAL PROJECTIVE TRANSFOEMATION

In view of the Berwald's covariant derivative [4],the Lie-derivative of a tensor field $T_j^i(x,\dot{x})$ and the connection parameter $G_{jk}^i(x,\dot{x})$ are given as under [7]:

$$\pounds_v T_j^i(x,\dot{x}) = T_{j(h)}^i v^h + (\dot{\partial}_h T_j^i) v_{(s)}^h \dot{x}^s + T_h^i v_{(j)}^h \tag{2.1}$$

and

$$\pounds_v G_{jk}^i(x,\dot{x}) = v_{(j)(k)}^i H_{jkh}^i v^h + G_{sjk}^i v_{(r)}^s \dot{x}^r, \tag{2.2}$$

where, $H_{jkh}^i(x,\dot{x})$ has been defined by (1.6).

We also have the following communication formula [7]:

$$\dot{\partial}_l (\pounds_v T_j^i) - \pounds_v (\dot{\partial}_l T_j^i) = 0 \tag{2.3}$$

$$\pounds_v T_{j(k)}^i - (\pounds_v T_j^i)_{(k)} = T_j^h \pounds_v G_{kh}^i - T_h^i \pounds_v G_{kj}^h - (\dot{\partial}_h T_j^i) \pounds_v G_{ks}^h \dot{x}^s \tag{2.4}$$

$$(\pounds_v G_{jh}^i)_{(k)} - (\pounds_v G_{kj}^i)_{(j)} = \pounds_v H_{hjk}^i + (\pounds_v G_{kl}^r) G_{rjh}^i \dot{x}^l - (\pounds_v G_{jl}^r) G_{rkh}^i \dot{x}^l . \tag{2.5}$$

We now give the following definitions which will be used in the later discussions.

DEFINITION (2.1):

A Finsler space F_n is said to admit an affine motion [3] provided there exists a vector $v^i(x)$ such that

$$\pounds_v G_{jk}^i(x,\dot{x}) = 0 . \tag{2.6}$$

DEFINITION (2.2) :

A Finsler space is said to be symmetric [1] if the Berwald's curvature tensor field $H_{hjk}^i(x,\dot{x})$ satisfies the relation

$$H_{hjk(m)}^i = 0 \tag{2.7}$$

the following relations also hold good in such a symmetric Finsler space

(a) $H_{jk(m)}^i = 0$,(b) $H_{j(m)}^i = 0$ and (c) $H_{(m)} = 0$. \hfill (2.8)

We now consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)dt \tag{2.9}$$

where, $v^i(x)$ stands for a non-zero contravariant vector field defined over the domain of the space and dt is an infinitesimal point constant .If such a transformation transforms the system of geodesics into the same system then such a transformation in F_n is termed as infinitesimal projective transformation .It has been mentioned in [3] that the necessary and sufficient condition in order that the infinitesimal point transformation given by (2.9) be an infinitesimal projective transformation is given by the following equation :

$$\pounds_v G_{jk}^i = \bar{G}_{jk}^i - G_{jk}^i = \delta_j^i p_k + \delta_k^i p_j - g_{jk} g^{il} d_l \tag{2.10}$$

here , $p_k(x,\dot{x})$ and $d_l(x,\dot{x})$ are covariant vectors and satisfy the following identities

(a) $\dot{\partial}_j p = p_j$, (b) $p_{hk} = \dot{\partial}_h \dot{\partial}_k p$, (c) $p_{hk} \dot{x}^h = p_k,$ \hfill (2.11)

(d) $p_{hk} \dot{x}^h \dot{x}^k = p$, (e) $\dot{\partial}_j d = d_j$, (f) $d_{hk} = \dot{\partial}_h \dot{\partial}_k d,$

(g) $d_{hk}\dot{x}^h = d_k$ and (h) $d_{hk}\dot{x}^h\dot{x}^k = d$

keeping in mind the formula (2.5) ,the Lie-derivative of $H^i_{hjk}(x,\dot{x})$ can be expressed in the following form

$$\pounds_v H^i_{hjk} = (\pounds_v G^i_{jh})_{(k)} - (\pounds_v G^i_{kh})_{(j)} + (\pounds_v G^r_{jl})\dot{x}^l G^i_{rhk} \quad . \tag{2.12}$$

Using (2.10) and (1.3) in (2.12) ,we get

$$\pounds_v H^i_{hjk} = \delta^i_j p_{h(k)} - \delta^i_k p_{h(j)} + \delta^i_h p_{j(k)} - \delta^i_h p_{k(j)} - g_{jh}g^{il}d_{l(k)} + g_{kh}g^{il}d_{l(j)} +$$

$$+ g_{kl}g^{rm}G^i_{rjh}d_m\dot{x}^l - g_{ij}g^{rm}G^i_{rkh}d_m\dot{x}^l \quad .$$

(2.13)

We now multiply (2.13) by $\dot{x}^h\dot{x}^j$ *and thereafter note (2.11) and the homogeneity property of* $H^i_{hjk}(x,\dot{x})$ *and get*

$$\pounds_v H^i_k = 2\dot{x}^i p_{(k)} - \delta^i_k p_{(j)}\dot{x}^j - \dot{x}^i p_{k(j)}\dot{x}^j - g_{jh}g^{il}d_{l(k)}\dot{x}^h\dot{x}^j + g_{kh}g^{il}d_{l(j)}\dot{x}^h\dot{x}^j \tag{2.14}$$

we now allow a contraction in (2.14) with respect to the indices i and k and thereafter use equations (1.7) and (2.11) and get

$$\pounds_v H = -p_{(j)}\dot{x}^j + \frac{1}{n-1}(d_{(j)}\dot{x}^j - g_{jh}g^{il}d_{l(i)}\dot{x}^h\dot{x}^j) \quad . \tag{2.15}$$

With the help of (2.15) and (2.14) ,we get

$$(\pounds_v H^i_k - \pounds_v H\delta^i_k) = 3\dot{x}^i p_{(k)} - \delta^i_k p_{(j)}\dot{x}^j - \dot{x}^i p_{k(j)}\dot{x}^j +$$

$$+ g_{kh}g^{il}d_{l(j)}\dot{x}^h\dot{x}^j - \frac{1}{n-1}\{d_k\dot{x}^i + (2-n)g_{jh}g^{il}d_{l(k)}\dot{x}^h\dot{x}^j\} \tag{2.16}$$

we now differentiate (2.16) partially with respect to $\dot{x}^r$ and thereafter allow a contraction in the resulting equation with respect to the indices i and r ,we get the following after making use of (1.7)and (2.11)

$$(\pounds_v \dot{\partial}_r H^r_k - \pounds_v \dot{\partial}_k H) = (3n+2)p_k - (n+3)p_{k(j)} + d_{k(j)}\dot{x}^j +$$

$$+ g_{kh}g^{rl}\dot{x}^h\{d_{rl(j)} + d_{lr}\} + (\frac{5-n}{n-1})d_k + 2\dot{x}^h\dot{x}^j\frac{C^l_{sr}}{g_{rs}}\{(\frac{2-n}{n-1})g_{rh}d_{l(k)} - g_{kh}d_{l(j)}\} \tag{2.17}$$

The underlined equation

$$\overline{G}^i(x,\dot{x}) = G^i(x,\dot{x}) - P(x,\dot{x})\dot{x}^i$$

(2.18)

represents the most general modification of the function G^i which will leave (2.18) unchanged .Thus ,we say that the equation (2.18) defines the projective change [4] of the function $G^i(x,\dot{x})$ .The tensor defined by

$$W^j_k(x,\dot{x}) = H^j_k - H\delta^j_k - \frac{1}{n+1}(\dot{\partial}_l H^j_k - \delta^j_k H)\dot{x}^l \tag{2.19}$$

is invariant under the projective change (2.18) and therefore it is regarded as projective deviation tensor .This deviation tensor also satisfies the following identities :

(a) $W^j_j = 0$, (b) $\dot{\partial}_K W^j_h \dot{x}^h = -W^j_k$ and

(2.20)

(c) $\dot{\partial}_i W^i_k = 0$.

The Lie-derivative of the projective deviation tensor $W_j^i(x,\dot{x})$ in view of (2.16) and (2.17) can be written in the following form

$$\pounds_v W_k^i = \frac{1}{n+1}[p_{(k)}\dot{x}^i + 2p_{k(j)}\dot{x}^i\dot{x}^j + \frac{4-n}{n-1}d_{(k)}\dot{x}^i - \dot{x}^i\{d_{k(j)}\dot{x}^j + g_{kh}g^{rl}\dot{x}^h(d_{rl(j)} +$$

$$+d_{l(r)}) + 2\dot{x}^h\dot{x}^j\frac{C_{sr}^l}{g_{rs}}(\frac{2-n}{n-1}g_{rh}d_{l(k)} - g_{kh}d_{l(j)})\}] -$$

$$-\delta_k^i p_{(j)}\dot{x}^j + g_{kh}g^{il}d_{l(j)}\dot{x}^h\dot{x}^j + \frac{2-n}{n-1}g_{jh}g^{il}d_{l(k)}\dot{x}^h\dot{x}^j . \tag{2.21}$$

We now apply the commutation formula given by (2.4) to the projective deviation tensor $W_j^i(x,\dot{x})$ and get

$$\pounds_v W_{j(r)}^i - (\pounds_v W_j^i)_{(r)} = W_j^h \pounds_v G_{rh}^i - W_h^i \pounds_v G_{jr}^h - (\dot{\partial}_h W_j^i)(\pounds_v G_{rs}^h)\dot{x}^s . \tag{2.22}$$

Using (2.2) and (2.3) in (2.22) ,we get

$$\pounds_v W_{j(r)}^i - (\pounds_v W_j^i)_{(r)} = W_j^h(\delta_r^i p_r - g_{rh}g^{il}d_l) - W_r^i p_j - 2W_j^i p_r + g^{hl}d_l\{W_h^i g_{jr} + \tag{2.23}$$

$$+(\dot{\partial}_h W_j^i)g_{rs}\dot{x}^s - (\dot{\partial}_r W_j^i)p\} .$$

We now allow a contraction in (2.23) with respect to the indices i and r and thereafter use (2.20) and get

$$\pounds_v W_{j(i)}^i - (\pounds_v W_j^i)_{(i)} = (n-2)W_j^h p_h - W_j^h d_h + g^{hl}d_l\{W_h^i g_{ji} + (\dot{\partial}_h W_j^i)g_{is}\dot{x}^s\} . \tag{2.24}$$

We now transvect (2.23) $\dot{x}^r$ and thereafter use (2.3) and (2.20) and get

$$\{\pounds_v W_{j(r)}^i - (\pounds_v W_j^i)_{(r)}\}\dot{x}^r = W_j^h\dot{x}^i p_h - 4W_j^i p - W_j^h g_{rh}g^{il}d_l\dot{x}^r + g^{hl}d_l\dot{x}^r + \tag{2.25}$$

$$+g^{hl}d_l\dot{x}^r\{W_h^i g_{jr} + (\dot{\partial}_h W_j^i)g_{rs}\dot{x}^s\} .$$

We now make an assumption that the space under consideration is symmetric one i.e. $W_{j(r)}^i = 0$ and as such under this assumption the equations (2.24) and (2.25) can alternatively be written in the following forms

$$(\pounds_v W_j^i)_{(r)} = (2-n)W_j^i p_r + W_j^i d_r - g^{hl}d_l\{W_h^i d_{ri} + (\dot{\partial}_h W_j^i)g_{is}\dot{x}^s\} \tag{2.26}$$

and

$$(\pounds_v W_j^i)_{(r)}\dot{x}^r = W_j^i p - W_j^h\dot{x}^i p_h + W_j^h g_{rh}g^{il}d_l\dot{x}^r - g_{hl}d^l\dot{x}^r\{W_h^i g_{jr} + (\dot{\partial}_h W_j^i)g_{rs}\dot{x}^s\} . \tag{2.27}$$

We now propose to eliminate the term $W_j^h p_h$ with the help of (2.26) and (2.27) and the result of elimination will give the following

$$M_j^i = [W_j^h d_h - g^{hl}d_l\{W_h^r g_{jr} + (\dot{\partial}_h W_j^i)g_{rs}\dot{x}^s\}]\dot{x}^i \tag{2.28}$$

where

$$M_j^i = (\pounds_v W_j^i)_{(r)}\dot{x}^r + (2-n)(\pounds_v W_j^i)_{(r)}\dot{x}^r . \tag{2.29}$$

At this stage ,if we now assume that the Finsler space F_n admits a projective motion which will be characterized by

$$\pounds_v G_{jk}^i = 0 .$$

(2.30)

Therefore ,in such a case ,with the help of (2.10) and (2.30) we shall easily arrive at the conclusion that the vectors $p(x,\dot{x})$ and $d(x,\dot{x})$ should separately vanish .

With the help of all these observations ,we can therefore state the following :

THEOREM(2.1):

In a Finsler space F_n equation (2.28) always holds provided the space under consideration admits a non – affine infinitesimal transformation such that the Berwald's covariant derivative of W_j^i remains an invariant .

THEOREM(2.2):

In a Finsler space F_n , $M_j^i = 0$ (where M_j^i has been given by (2.29)) provided the space under consideration admits an affine infinitesimal transformation such that the Berwald's covariant derivative of W_j^i remains an invariant .

THEOREM (2.3):

In a Finsler space F_n,(2.28) necessarily holds provided the space under consideration is symmetric one and it admits a non –affine infinitesimal transformation .

THEOREM (2.4):

In a Finsler space F_n,(2.26) necessarily holds provided the space under consideration is symmetric .

3.INFINITESIMAL SPECIAL PROJECTIVE TRANSFORMATION

In view of the projective covariant derivative as has been given by (1.8) and the projective connection coefficient $\Pi_{jk}^i(x,\dot{x})$ as has been given by (1.9) ,the Lie-derivatives of an arbitrary tensor $T_j^i(x,\dot{x})$ and the projective connection coefficient are respectively given by

$$£_v T_j^i(x,\dot{x}) = T_{j((r))}^i v^r + (\dot{\partial}_s T_j^i)v_{((r))}^s \dot{x}^r - T_j^r v_{((r))}^i + T_r^i v_{((j))}^r \qquad (3.1)$$

and

$$£_v \Pi_{mk}^i(x,\dot{x}) = v_{((m))((k))}^i + Q_{mkr}^i v^r + (\dot{\partial}_r \Pi_{mk}^i)v_{((s))}^r \dot{x}^s \qquad (3.2)$$

.

In between the operators $£_v, \dot{\partial}$ and $(( ))$, we have the following commutation formulae:

(a) $$\dot{\partial}_\rho(£_v T_j^i) - £_v(\dot{\partial}_\rho T_j^i) = 0 \qquad ,$$

(3.3)

(b) $$(£_v T_j^i)_{((r))} - £_v T_{j((r))}^i = T_j^i £_v \Pi_{lr}^l - T_l^i £_v \Pi_{rj}^l - (\dot{\partial}_l T_j^i)£_v \Pi_{rm}^l \dot{x}^m$$

and (c) $$(£_v \Pi_{hj}^i)_{((k))} - (£_v \Pi_{hk}^i)_{((j))} = £_v Q_{hkj}^i + (£_v \Pi_{jb}^l)\Pi_{hkl}^i \dot{x}^b + (£_v \Pi_{kb}^l)\Pi_{jhl}^i \dot{x}^b$$.

In order that the infinitesimal point transformation given by (2.9) may define an infinitesimal special projective transformation ,it is necessary and sufficient that [3]:

$$£_v \Pi_{jk}^i = \bar{\Pi}_{jk}^i - \Pi_{jk}^i = \delta_j^i b_k + \delta_k^i b_j - g_{jk} g^{il} c_l \qquad (3.4)$$

where , $b_k(x,\dot{x})$ and $c_l(x,\dot{x})$ are covariant vectors and they satisfy the following relations :

(a) $\dot{\partial}_j b = b_j$, (b) $b_{hk} = \dot{\partial}_h \dot{\partial}_k b$, (c) $b_{hk}\dot{x}^h = b_k$, (3.5)

(d) $b_{hk}\dot{x}^h \dot{x}^k = b$, (e) $\dot{\partial}_j c = c_j$, (f) $c_{jk} = \dot{\partial}_j \dot{\partial}_k c$,

(g) $c_{hk}\dot{x}^h = c_k$ and (h) $c_{hk}\dot{x}^h \dot{x}^k = c$.

Using (3.4) ,(3.5) and the commutation formula given by (3.3 c) , the Lie-derivative of the projective entity $Q_{hjk}^i(x,\dot{x})$ can be written in the following form

$$£_v Q_{hjk}^i = \delta_j^i b_{h((k))} + \delta_h^i b_{j((k))} - g_{jh} g^{il} c_{l((k))} - g_{jh((k))} g^{il} c_l$$
$$- g_{jh} g_{((k))}^{il} c_l - \delta_k^i b_{h((j))} - \delta_h^i b_{k((j))} + g_{kh} g^{il} c_{l((j))} +$$

$$+g_{kh((j))}g^{il}c_l + g_{kh}g^{il}_{((j))}c_l - \delta^r_k b\Pi^i_{rjh} + g_{kl}g^{rm}c_m\Pi^i_{rjh}\dot{x}^l +$$

$$+b\delta^r_j\Pi^i_{rkh} - g_{jl}g^{rm}c_m\dot{x}^l\Pi^i_{rhk} \tag{3.6}$$

we now transvect (3.6) by $\dot{x}^h\dot{x}^j$ and therefore use (1.12) and (1.13) together and get

$$£_v Q^i_k = 2\dot{x}^i b_{((k))} - \delta^i_k b_{((j))}\dot{x}^j + \dot{x}^h\dot{x}^j[g_{kh}\{g^{il}_{((j))}c_l + g^{il}c_{l((j))}\} - \tag{3.7}$$

$$-g_{jh}\{g^{il}_{((k))}c_l + g^{il}c_{l((k))}\} - g_{jh((k))}g^{il}c_l].$$

We now allow a contraction in (3.6) with respect to the indices i and k and thereafter transvecting the equation thus obtained by $\dot{x}^h\dot{x}^j$, we get

$$£_v Q_{hj}\dot{x}^h\dot{x}^j = (1-n)b_{((j))}\dot{x}^j + c_{((j))}\dot{x}^j + g^{il}c_l\dot{x}^h\dot{x}^j\{g_{ih((j))} - g_{jh((i))}\} -$$

$$-g_{jh}\dot{x}^h\dot{x}^j\{g^{il}c_{l((i))} + g^{il}_{((i))}c_l\} + g_{ih}g^{il}_{((j))}c_l\dot{x}^h\dot{x}^j \tag{3.8}$$

where we have taken into account (1.12).

We now eliminate $b_{((j))}\dot{x}^j$ using (3.7) and (3.8) and get

$$L^i_k(x,\dot{x}) = 2(1-n)b_{((k))}\dot{x}^i - b_{k((j))}\dot{x}^i\dot{x}^j + g_{kh}\dot{x}^h\dot{x}^j\{g^{il}_{((j))}c_l + g^{il}c_{l((j))}\} -$$

$$-g^{il}_{((k))}c_l - g^{il}c_{l((k))} + c_{((j))}\dot{x}^j\delta^i_k + g^{il}c_l\dot{x}^h\dot{x}^j\delta^i_k\{g_{ih((j))} - g_{jh((i))}\} -$$

$$-g_{jh}\dot{x}^h\dot{x}^j\delta^i_k\{g^{il}c_{l((i))} - g^{il}_{((i))}c_l\} + g_{ih}g^{il}_{((j))}c_l\dot{x}^h\dot{x}^j\delta^i_k \tag{3.9}$$

where,

$$L^i_k \stackrel{def}{=} £_v Q^i_k + \delta^i_k £_v Q_{hj}\dot{x}^h\dot{x}^j \tag{3.10}$$

We now apply the commutation formula (3.36) to the projective deviation tensor $W^i_j(x,\dot{x})$ and thereafter use (3.4) and (3.5) and get

$$(£_v W^i_j)_{((r))} - £_v W^i_{j((r))} = W^l_j\delta^i_r b_l - W^l_j g_{rl}g^{ip}c_p - W^i_r b_j + W^i_l g_{rj}g^{lp}c_p -$$

$$-(\dot{\partial}_r W^i_j)b - 2W^i_j b_r - (\dot{\partial}_r W^i_j)g_{lm}g^{lp}c_p\dot{x}^m$$

(3.11)

we now allow a contraction in (3.11) with respect to the indices i and r and get

$$(£_v W^i_j)_{((i))} - £_v W^i_{j((i))} = (n-2)W^l_j b_l - W^l_j c_l + g^{lp}c_p\{W^i_l g_{ij} - (\dot{\partial}_l W^i_j)g_{im}\dot{x}^m\} \tag{3.12}$$

we now transvect (3.11) by $\dot{x}^r$ and thereafter use (3.5) and get

$$\{(£_v W^i_j)_{((i))} - £_v W^i_{j((r))}\}\dot{x}^r = W^l_j b_l\dot{x}^i - 4W^i_j b - W^l_j g_{rl}g^{ip}c_p\dot{x}^r +$$

$$+W^i_l g_{rj}g^{lp}c_p\dot{x}^r - (\dot{\partial}_l W^i_j)g_{rm}g^{lp}c_p\dot{x}^r\dot{x}^m$$

(3.13)

We now make the supposition that the infinitesimal special projective transformation given by (3.4) leaves invariant the projective covariant derivative of the projective deviation tensor, i.e.

$$£_v W^i_{j((r))} = 0 \tag{3.14}$$

As a result of this supposition, the equation (3.12) and (3.13) can respectively be expressed in the following alternative form

$$(\pounds_v W_j^i)_{((i))} = (n-2)W_j^l b_l - W_j^l c_l + g^{lp}c_p\{g_{ij}W_l^i - (\dot{\partial}_l W_j^i)g_{im}\dot{x}^m\} \tag{3.15}$$

and

$$(\pounds_v W_j^i)_{((r))} = W_j^l b_l \dot{x}^i - 4W_j^i b - W_j^l g_{rl}g^{ip}c_p\dot{x}^r +$$
$$+W_l^i g_{rj}g^{ip}c_p\dot{x}^r - (\dot{\partial}_l W_j^i)g_{rm}g^{lp}c_p\dot{x}^r\dot{x}^m \tag{3.16}$$

.

We now propose to eliminate $W_j^l b_l$ with the help of (3.15) and (3.16) ,the result of elimination gives the following :

$$B_j^i(x,\dot{x}) = \dot{x}^i[-W_j^l c_l + g^{lp}c_p\{W_l^k g_{kj} - (\dot{\partial}_l W_j^r)g_{rm}\dot{x}^m\}] + (n-2)[4W_j^i b +$$
$$+W_j^l g_{rl}g^{ip}c_p\dot{x}^r - W_l^i g_{rj}g^{lp}c_p\dot{x}^r + (\dot{\partial}_l W_j^i)g_{rm}g^{lp}c_p\dot{x}^r\dot{x}^m] \tag{3.17}$$

where,

$$B_j^i(x,\dot{x}) \overset{def}{=} (\pounds_v W_j^i)_{((r))}\dot{x}^r - (n-2)(\pounds_v W_j^i)_{((r))}\dot{x}^r. \tag{3.18}$$

In order that the space under consideration may admit a special projective affine motion ,we will always have

$$\pounds_v \Pi_{jh}^i = 0$$

.

(3.19)

Using (3.4) and (3.19) ,we shall easily arrive at the conclusion that the vectors $b(x,\dot{x})$ and $c(x,\dot{x})$ must separately vanish.

In the light of all these observations ,we can therefore state :

THEOREM (3.1): In a Finsler space F_n ,the equation (3.17) always holds provided the space under consideration admits a non-affine infinitesimal special projective transformation such that the projective covariant derivative of projective deviation tensor W_j^i remains an invariant .

THEOREM(3.2): In a Finsler space F_n , $B_j^i(x,\dot{x})$ given by (3.18) always vanishes provided the space under consideration admits an affine infinitesimal special projective transformation such that the projective covariant derivative of the projective deviation tensor W_j^i remains an invariant .

If the Finsler space F_n under consideration be assumed to be symmetric one i.e. $W_{j((r))}^i = 0$,then under such an assumption the equation (3.14) will always hold.Therefore ,we can state :

Theorem (3.3): In a symmetric Finsler space F_n ,equation (3.17) always holds provided the space under consideration admits a non-affine infinitesimal special projective transformation characterized by (3.4) .

THEOREM(3.4): In a symmetric Finsler space F_n , B_j^i characterized by (3.18) always vanishes provided the space under consideration admits an affine infinitesimal special projective transformation .

CONCLUSION :

The present communication has been divided into three sections of which the first section is introductory .The second section deals with non-affine infinitesimal transformations . In this section ,we have derived conditions which will hold when the space under consideration admits non-affine as well as an affine infinitesimal transformation and in the sequel have established the conditions which will hold when the space is symmetric and it admits an affine as well as non-affine infinitesimal transformation .The third section deals with infinitesimal special projective transformation .Like the previous section ,in this section

too we have established the conditions which will hold when the space under consideration is symmetric and it admits a non-affine as well as an affine infinitesimal special projective transformation.

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