International Journal of Scientific Research in Science and Technology
Available online at : www.ijsrst.com
Print ISSN: 2395-6011 | Online ISSN: 2395-602X
doi : https://doi.org/10.32628/IJSRST

# General Theory of The Matrix Differential Operator and Spectral Theorem

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ARTICLEINFO

**Article History:**
Accepted: 10 Nov 2023
Published: 24 Nov 2023

**Publication Issue**
Volume 10, Issue 6
November-December-2023

**Page Number**
517-520

ABSTRACT

In this present paper, the general theory of the matrix differential operator and spectral theorem has been explained. The theory of eigenfunction expansions associated with the second-order differential equations and their spectral behavior has been also presented in this paper.

Keywords : Matrix differential operator, Spectral theorem, Convergence theorem.

## I. INTRODUCTION

Titchmarsh [1] in 1944 discussed the finite case of the simultaneous system of two first-order linear differential equations and he considered the extension to the infinite case in 1941.

Conte & Sangren [2] discussed two first-order equations in 1953 and 1954. Since then, Roes and Sangren have worked as the same problem in a series of papers. Their methods are those of Titchmarsh's complex variable methods.

## DIFFERENTIAL EQUATIONS OF SECOND ORDER

Hilbert [3] initiated the discussion of a pair of simultaneous differential equations of second order. He considered the systems

$$L_1(u_1u_2) = -\lambda\{k_{i1}(x)u_1 + k_{i2}u_2\}, \qquad i = 1,2 \;\ldots\ldots\ldots (1)$$

where

$$L_1(u_1u_2) = \frac{1}{2}\left\{\frac{d}{dx}\frac{\partial Q}{\partial u_i} - \frac{\partial Q}{\partial u_i}\right\}, i = 1, 2 \ldots\ldots\ldots.$$

Q being the homogenous quadratic expression

$$Q(u_1'u_2'u_1u_2) = \sum p_{ij}(x)u_i'u_j' + 2\sum q_{ij}\,u_i'u_j + \sum r_{ij}u_iu_j$$

where

$$p_{ij} = p_{ji}\; and\; r_{ij} = r_{ji}, i, j = 1,2$$

with the boundary

$$u_1(a) = u_2(a) = 0\,;\; u_1(b) = u_2(b)$$

Eastham [4] obtained separation and oscillation theorems for the system.

$$\left.\begin{aligned} y'' + p(x)\,y &= q(x)\,z, \\ z'' + p(x)\,z &= r(x)\,y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

and gave the interesting dynamical interpretation of the problem.

Kamke [38] has mentioned the self-adjoint system.

$$y'' + \lambda\,(\lambda_{11}\,y + \lambda_{12}\,z) = 0$$

$$z'' + \lambda\,(\lambda_{21}\,z + \lambda_{22}\,y) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

with suitable boundary conditions and stated formulae giving the eigenvalue $A_p$ : p = 1,2,3………

Lidskil [5] considered the system

$$-\,y'' + [(x)\,y = \lambda y, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

y = {$y_1$, $y_2$, ……, $y_n$} is a vector and p is a symmetric matrix of real functions. He obtained sets of general conditions under which, for complex values of λ, (i) the given system has usually a linearly independent $L^2$ solutions and (ii) the system has the solutions belonging to $L^2$ [0, ∞).

## GENERAL THEORY OF THE MATRIX DIFFERENTIAL OPERATOR

Kamke [6] has developed a general theory of the matrix differential operator

$$L = p_0(x)\left(\frac{d}{dx}\right)^n + p_1(x)\left(\frac{d}{dx}\right)^{n-1} + \cdots + p_n(x) \ldots\ldots \quad (5)$$

in a finite interval (a,b) with matrix coefficients

$$p_k(x) = \left(p_{ij}(x)\right), i, j = 1,2,3 \ldots.., n$$

on vector functions

$$U\,(x) = [U_1(x), U_2(x), \ldots\ldots, U_n(x)]$$

Coddington and Levinson [101] have considered the differential system.

$$p_k(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n(x) = 1x, \ldots.\ \ldots\ldots\ldots\ldots \quad (6)$$

with boundary conditions

$$M_1x^{n-1}(a) + \ \ldots + M_n\,x(a) + N_1x^{n-1}(b) + \ \ldots + N_n\,x(a) = 0,$$

where x is a vector with r components, $p_k$; k = 1, 2, ….., n, by - r matrices of class $c^{n-k}$ [a,b], and det. $p_0$ (t), 0, $M_k$, $N_k$ are matrices of n rows and r columns.

Pandey [7] has devoted a whole chapter on the discussion of the system of equations of Kodaira type, confining the discussion wholly to the finite case.

Chakravarty [8-11] has discussed the system

$$(L - \lambda)\,y = 0, \ldots. \quad \ldots.. \quad (7)$$

Where L stands for the matrix operator

$$L = \begin{pmatrix} p(x) & \dfrac{d^2}{dx^2} + q(x) \\ \dfrac{d^2}{dx^2} + q(x) & x(x) \end{pmatrix}$$

and y = {u,v} is a vector having the components u and v.

Roos [12] has considered the nth order linear vector differential equation.

L [u] = λM [x], ………………………….. (8)

together with the boundary conditions

$$M_\alpha(x) = \sum_{\beta=1}^{n} \{M_{\alpha\beta} u^{\beta-1}(a) + M_{\alpha n^+\beta} u^{\beta-1}(b)\} = 0$$

(α = 1, 2, …., n)

Where,

U(t) = [$u_\sigma$(t), = 1, 2,….., h]

is a n-dimensional vector function and L and M are defined as

$$L\ [u] = p_n(t)u^{(n)} + p_{n-1}(t)u^{(n-1)} + \cdots p_o(t)u,$$
$$M\ [x] = Q_n(t)u^{(n)} + Q_{r-1}(t)u^{(r-1)} + \cdots Q_o(t)u,$$

Where 0 ≤ r < n and n x n matrix functions

$p_0$(t),…….., $p_n$(t), $Q_0$ (t), ………$Q_r$(t) are continuous on [a,b] such that $Q_r$ (t) ≠ 0 and $p_n$(t) is non-singular in this interval.

Bhagat [13] has considered the differential system.

(L- λF)ϕ = 0 …………………………….. (9)

over a finite interval [a,b], where L stands for the matrix operator given by

$$L \equiv \begin{pmatrix} \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & \frac{d}{dx}\left(q_0 \frac{d}{dx}\right) + q_1(x) \end{pmatrix}$$

F the symmetric matrix

F ≡ ($F_{ij}$(x)),

and ϕ a vector represented by a column matrix

$$\phi = \begin{pmatrix} U \\ V \end{pmatrix}$$

He has taken the boundary conditions at x = a and x = b respectively as

$$M(a,\phi) = p_0\ (a)\left[a_{j1}u(a) + a_{j2}u'(a)\right] + q_0\ (a)\left[a_{j3}v(a) + a_{j4}v'(a)\right] = 0$$
$$N(b,\phi) = p_0\ (b)\left[b_{j1}u(b) + a_{j2}u'(b)\right] + q_0\ (a)\left[b_{j3}v(b) + b_{j4}v'(b)\right] = 0$$

(j = 1,2)……. (10)

## II. CONCLUSION

The value of λ for which the system (Equation 9) has a non-trivial solution satisfying the boundary conditions (Equation 10) is an eigenvalue and the corresponding vector solution is an eigenvector.

Here, discussed same of the properties of the boundary value problem (Equation 9) and (Equation 10) and established the expansion theorem.

In [Equation 4], then extended some of the results is the interval [0, ∞]. The nature of singular surface has been discussed and proved that there exist at least two solutions $\psi_1$ and $\psi_2$ of the system (Equation 9) such $\psi_r^T F \psi_r$ that belongs to L [0, ∞]. The existence of Green's matrix in [0, ∞] and its uniqueness under certain conditions have been proved (see [Equation 4]). For a special case of the problem (Equation 9) where $p_0(x) = q_0(x) = 1$, F is a unit matrix and λ replaced by -λ, proved a spectral theorem in [Equation 5].

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**Cite this article as :**

Rajeev Ranjan, "General Theory of The Matrix Differential Operator and Spectral Theorem", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 10 Issue 6, pp. 517-520, November-December 2023.

Journal URL : https://ijsrst.com/IJSRST52310676