International Journal of Scientific Research in Science and Technology
Available online at : www.ijsrst.com
Print ISSN: 2395-6011 | Online ISSN: 2395-602X
doi : https://doi.org/10.32628/IJSRST52310634

Eigenfunction Expansion and Spectral Theorem

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ARTICLEINFO

Article History:

Accepted: 05 Jan 2024
Published: 22 Jan 2024

Publication Issue :
Volume 11, Issue 1
January-February-2024

Page Number :
280-285

ABSTRACT

In this present paper, the theory of eigenfunction expansions associated with the second-order differential equations and their spectral behavior.

Keywords: Matrix differential operator, Spectral theorem, Convergence theorem.

I. INTRODUCTION

We consider the differential system

$$(M+\lambda)\,\phi = 0; \qquad 0 \le \times < \infty \tag{1}$$

where M stands for the matrix differential operator given by

$$M \equiv \begin{pmatrix} \frac{d^2}{dx^2} - p(x) & r\,(x) \\ r(x) & \frac{d^2}{dx^2} - q(x) \end{pmatrix}$$

ϕ is a vector represented by a column matrix

$$\phi = \begin{pmatrix} \frac{u}{v} \end{pmatrix}$$

and λ is a real parameter.

We assume the following conditions to be satisfied:

(i) p(x), q(x), r(x) is all real - valued and continuous functions in $0 \le x \le \infty$

(ii) p(x), q(x), r(x) is all $L\,[0,\infty)$. We suppose that any solution $\phi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$

of the system satisfies the two linearly independent boundary conditions at x=0, viz :

$$S_{jl}u(0) + a_{j2}u'(0) + a_{j3}v(0)a_{j4}v'(0) = 0, (j = 1, 2) \tag{2}$$

where

(a) $a_{jk}\ \{j = 1,2; k = 1, 2,3,4\}$ are real-valued constants;

(b) the set $\{a_{lk}\}$ is linearly independent of the set $\{a_{2k}\}$;

(c) $a_{14}a_{23} - a_{24}a_{13} + a_{12}a_{21} - a_{11}a_{22} = 0$ (3)

Following Bhagat [3], the bilinear concomitant $[\phi\theta]$ of two vector $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and $\theta = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$ is defined by

$$\phi = \phi_1'\theta_1\phi_1' + \phi_1'\theta_2 - \phi_2\,\theta_2'$$

If ϕ and θ are any two solution of the system (1) for the same value of λ, then $[\phi\theta]$ is a function of λ, then $[\phi\theta]$ is a function of λ, real for real λ (See Bhagat (3)).

Let

$$\phi_j(x_2\lambda) = \phi_j(0|x_2\lambda) = \begin{pmatrix}u_j(0|x_2\lambda)\\ v_j(0|x_2\lambda)\end{pmatrix} \qquad (j = 1, 2)$$

by the boundary-condition vector then (2) and (3) can be written as

$$[\phi(x_2\lambda)\,\phi_j(0|x,\lambda)] = 0 \quad (j = 1,2) \qquad (4)$$

and

$$[\phi_1\phi_2] = 0 \qquad (5)$$

The vectors

$$\theta_k(x_2\lambda) = \theta_k(0|x_2\lambda) = \begin{pmatrix}x_k(0|x_2\lambda)\\ y_k(0|x_2\lambda)\end{pmatrix} \qquad (k = 1,2)$$

which take real constant values (independent of λ) at x=0 is defined by the relations

$$\phi_j\theta_k = \delta_{jk} = [\theta_1\theta_2] = 0\ (1 \le j,k \le 2) \qquad (6)$$

(See Bhagat [5]).

It has been shown by Bhagat in [4] that the system (1.) has at least a pair of solutions belonging to $L^2[0.\infty]$ which are given by

$$\psi_r(x,\lambda) = \theta_r(x,\lambda) + \sum_{s=1}^{2} m_{rs}((\lambda)\,\phi_s(x,\lambda), (r = 1,2) \qquad (7)$$

The $m_{rs}(\lambda)$ $(1\le r,\ s \le 2)$ are analytic functions of λ regular in either of the half plane im $\lambda > 0$ and im $\lambda < 0_2$ and $\overline{m_{rs}(\lambda)} = m_{rs}(\bar{\lambda})$. It is also proved that

$$[\phi_j(0|x_2\lambda)\psi_r(x,\lambda)] = \delta_{jr^2}(1 \le j,r \le 2) \qquad (8)$$

(See Bhagat [4]).

2. THE GREEN'S MATRIX

$$G(x,y;\lambda) = \begin{pmatrix}G_{11} & G_{21}\\ G_{12} & G_{22}\end{pmatrix}$$

for the system (1) is given by

$$G(x,y;\lambda) = \begin{pmatrix}\psi_{11}(x,\lambda) & \psi_{21}(x,\lambda)\\ \psi_{12}(x,\lambda) & \psi_{22}(x,\lambda)\end{pmatrix}\begin{pmatrix}u_1(y,\lambda) & v_1(y,\lambda)\\ u_2(y,\lambda) & v_2(y,\lambda)\end{pmatrix} y \in [0,x]$$

$$= \begin{pmatrix}u_1(y,\lambda) & u_2(y,\lambda)\\ v_1(y,\lambda) & v_2(y,\lambda)\end{pmatrix}\begin{pmatrix}\psi_{11}(y,\lambda) & \psi_{12}(y,\lambda)\\ \psi_{21}(y,\lambda) & \psi_{21}(y,\lambda)\end{pmatrix}; y \in (x,\infty) \qquad (9)$$

We shall use the notations and results of Bhagat [3-9]

3. INTEGRAL EQUATIONS

$\phi_j(x,\lambda)$ $(j = 11,2)$ satisfy the system of integral equations

$u_j(x,\lambda) = u_j(0)\cos\mu\, x + \frac{1}{\mu}u_j'(0)\sin\mu x +$

$$+\frac{1}{\mu}\int_0^x \{p(y)\mathrm{u_j}(y,\lambda) - r\,(y)\;\mathrm{v_j}(y,\lambda)\}\sin\mu\,(x-y)\;\mathrm{dy},$$

$$\mathrm{v_j}(x,\lambda) = \mathrm{v_j}(0)\cos\mu\,x + \frac{1}{\mu}\mathrm{v}_j'\,(0)\sin\mu x +$$

$$+\frac{1}{\mu}\int_0^x \{q(y)\mathrm{v_j}(y,\lambda) - r\,(y)\;\mathrm{u_j}(y,\lambda)\}\sin\mu\,(x-y)\;\mathrm{dy},$$

(j = 1, 2) (10)

where $\lambda = \mu^2$.

We have from [5] for large x, if $\mu = \sigma + \mathrm{it}$, $\mathrm{t} \geq 0$ and $|\mu| \geq \rho >)$

$$\left.\begin{aligned} &\mathrm{u_j}(y,\lambda), \mathrm{v_j}(x,\lambda) = 0\,(e^{tx}), (j=1,2), \\ &\mathrm{u_j}(x,\lambda) = e^{-1\mu x}\{M_{j1}(\lambda) + 0(1)\}, \end{aligned}\right\} \quad (11)$$

(j = 1, 2) (12)

$$\mathrm{v_j}(x,\lambda) = e^{-1\mu x}\{M_{j2}(\lambda) + 0(1)\},$$

where

$$\left.\begin{aligned} M_{j1}(\lambda) &= \tfrac{1}{2}\mathrm{u_j}(0) - \tfrac{1}{2i\mu}\,\mathrm{u}_j'(0) - \\ &-\frac{1}{2i\mu}\int_0^\infty e^{1/\mu y}\{\mathrm{p}\,(\mathrm{y})\mathrm{u_j}(y,\lambda) - r\,(y)\mathrm{v_j}(y,\lambda)\}\,dy, \\ M_{j2}(\lambda) &= \frac{1}{2}\mathrm{v_j}(0) - \frac{1}{2i\mu}\,\mathrm{v}_j'(0) - \\ &-\frac{1}{2i\mu}\int_0^\infty e^{1/\mu y}\{\mathrm{q}\,(\mathrm{y})\mathrm{v_j}(y,\lambda) - r\,(y)\mathrm{u_j}(y,\lambda)\}\,dy \end{aligned}\right\}$$

(j=1, 2) (13)

Also from [6; , ξ2) for $|\mu| \geq |\mu_o|$

$$\left.\begin{aligned} \mathrm{u_j}(\mathrm{x},\lambda) &= u_j(0)\cos\mu x + 0\left\{\frac{e^{|tx|}}{|\mu|}\right\}. \\ \mathrm{v_j}(\mathrm{x},\lambda) &= \mathrm{v}_j(0)\cos\mu x + 0\left\{\frac{e^{|tx|}}{|\mu|}\right\}. \end{aligned}\right\} \quad (\mathrm{j}=1,2) \quad (14)$$

4. SPECIAL SOLUTIONS

In this section we obtain two independent solutions of (1) which are small when imaginary part of λ is large and positive.

Consider the system of integral equations

$$\mathrm{X_j}\,(\mathrm{x}) = e^{|\mu x|} + \frac{1}{2i\mu}\int_0^x e^{1\mu(x-y)}\{\mathrm{p}\,(\mathrm{y})\mathrm{X_j}(y) - r\,(y)\mathrm{Y}_j(y)\}\,dy +$$

$$+\frac{1}{2i\mu}\int_x^\infty e^{1\mu(\mathrm{y-x})}\{\mathrm{p}\,(\mathrm{y})\,\mathrm{X_j}\,(y) - r\,(y)\mathrm{Y}_j(y)\}\,dy,$$

$$\mathrm{Y_j}\ (\mathrm{x}) = e^{|\mu x|} + \frac{1}{2i\mu}\int_0^x e^{1\mu(x-y)}\{\mathrm{q}\ (y)\mathrm{Y_j}(y) - r\ (y)\mathrm{X}_j(y)\}\, dy +$$

$$+\frac{1}{2i\mu}\int_x^{\infty} e^{1\mu(y-x)}\{\mathrm{q}\ (y)\ \mathrm{X_j}\ (y) - r\ (y)\mathrm{X}_j(y)\}\, dy,$$

$$(\mathrm{j} = 1, 2) \qquad (15)$$

where $\lambda = \mu^2$

Differentiating (15) twice it can be verified (formally) that $\beta_j(\mathrm{x}) = \begin{pmatrix}\mathrm{X_j(x)}\\ \mathrm{Y_j(x)}\end{pmatrix}$, (j = 1, 2) satisfy (1).

The solutions of (15) can be obtained by the method of successive approximation as follows :

Let

$$\mathrm{X_{j1}}(\mathrm{x}) = e^{\mathrm{i\mu x}}, \mathrm{Y_{j1}}(\mathrm{x})\ e^{\mathrm{i\mu x}}, \quad (\mathrm{j} = 1, 2) \qquad (16)$$

and for $n \geq 1$

$$X_{jn+1}(x) = e^{1\mu x}\frac{1}{2i\mu}\int_{\infty}^{x} e^{1\mu(x-y)}\{\mathrm{p}\ (y)\mathrm{X_{jn}}(y) - r\ (y)\mathrm{Y}_{jn}(y)\}\, dy +$$

$$+\frac{1}{2i\mu}\int_x^{\infty} e^{1\mu(y-x)}\{\mathrm{p}\ (y)\mathrm{X_{jn}}(y) - r\ (y)\mathrm{Y}_{jn}(y)\}\, dy,$$

$$Y_{jn+1}(x) = e^{1\mu x}\frac{1}{2i\mu}\int_{\infty}^{x} e^{1\mu(x-y)}\{\mathrm{q}\ (y)\mathrm{X_{jn}}(y) - r\ (y)\mathrm{X}_{jn}(y)\}\, dy +$$

$$+\frac{1}{2i\mu}\int_x^{\infty} e^{1\mu(y-x)}\{\mathrm{q}\ (y)\mathrm{Y_{jn}}(y) - r\ (y)\mathrm{X}_{jn}(y)\}\, dy,$$

$$(\mathrm{j} = 1, 2) \qquad (17)$$

Since p(x), q(x), r(x) are all L [0, ∞], so we suppose that

$$\mathrm{J} = \mathrm{Max}\left\{\int_0^{\infty} |\mathrm{p(x)}|\mathrm{dx},\ \int_0^{\infty} |\mathrm{q(x)}|\mathrm{dx},\ \int_0^{\infty} |\mathrm{r(x)}|\mathrm{dx}\right\} \qquad (18)$$

Then

$$\mathrm{X_{j2}}\ (\mathrm{x}) - \mathrm{X_{jl}}(\mathrm{x}) = \frac{e^{1\mu\mathrm{x}}}{2i\mu}\left[\int_0^{\mathrm{x}} \{\mathrm{p}\ (\mathrm{y}) - \mathrm{r}\ (\mathrm{y})\}\mathrm{dy} + \int_x^{\infty} \{\mathrm{p}\ (\mathrm{y}) - \mathrm{r}\ (\mathrm{y})\}\mathrm{e}^{2\mathrm{i\mu}\ (\mathrm{y}-\mathrm{x})}\mathrm{dy}\right]$$

or , $\left|\mathrm{X_{j2}}\ (\mathrm{x}) - \mathrm{X_{jl}}(\mathrm{x})\right| \leq \frac{e^{-tx}}{|\mu|}\mathrm{J}$, (j = 1, 2) (19)

Similarly

$\left|\left|\mathrm{Y_{j2}}\ (\mathrm{x}) - \mathrm{Y_{jl}}(\mathrm{x})\right| \leq \frac{e^{-\mathrm{tx}}}{|\mu|}\mathrm{J},\right|$ (j = 1, 2) (20)

Hence by using (4.4.5) and (4.4.6) we have

$\left|\mathrm{X_{j3}}\ (\mathrm{x}) - \mathrm{X_{j2}}(\mathrm{x})\right| \leq \frac{e^{-tx}}{|\mu|^2}\mathrm{J}^2$ (j = 1, 2) (21)

and

$\left|\mathrm{Y_{j3}}\ (\mathrm{x}) - \mathrm{X_{j2}}(\mathrm{x})\right| \leq \frac{e^{-tx}}{|\mu|^2}\mathrm{J}^2$, (J = 1, 2) (22)

and so on.

Therefore, it follows that if $|\mu| > \mathrm{J}$, the series

$$\sum_{n=1}^{\infty}(X_{jn+1}(x) - X_{jn}(x))$$

and

$$\sum_{n=1}^{\infty}(X_{jn+1}(x) - X_{jn}(x))$$

are convergent.

Let $X_j((x) = \lim_{n\to\infty} X_{jn}(x)$ and $Y_j(x) \lim_{n\to\infty} Y_{jn}(x)$

Now for every n

$$|X_{jn}(x)| \le |X_{j1}(x)| + |X_{j2}(x) - X_{j1}(x)| + \ldots. + |X_{jn}(x) - xX_{jn-1}(x)|$$
$$\le e^{-tx}\{1 + (J/|\mu|) + \cdots + (J/|\mu|^{N-1}\}$$
$$= e^{-tx}\frac{[1-(J/|\mu|)^N]}{[1-(J/|\mu|)]} \le e^{-tx}/\left(1-\frac{J}{|\mu|}\right)$$

so for n ⟶ ∞

$$|X_j(x)| = \lim_{n\to\infty} X_{jn}(x) / \le \frac{e^{-tx}}{\left(1-\frac{J}{|\mu|}\right)}, \qquad (j=1,2) \qquad (23)$$

Similarly

$$|Y(x)| = \lim_{n\to\infty} |Y_{jn}(x) / \le \frac{e^{-tx}}{\left(1-\frac{J}{|\mu|}\right)}, \qquad (j=1,2) \qquad (24)$$

Therefore, by dominated convergence, it follows that the limit operation can be taken under the integral sign and that $\beta_j(x)(j = 1,2)$ satisfy the equations (15) and hence (1)

Now for a fixed μ or μ in the bounded part of region $|\mu| > J$, (15) gives

$$X_j(x) = e^{i\mu x} + \frac{e^{i\mu x}}{2i\mu}\int_0^{\infty}\{p(y)X_j(y) - r(y)Y_j(y)\}e^{-1\mu y}\,dy -$$
$$-\frac{e^{i\mu x}}{2i\mu}\int_x^{\infty}\{p(y)X_j(y) - r(y)Y_j(y)\}e^{-1\mu y}\,dy +$$
$$+\frac{e^{i\mu x}}{2i\mu}\int_x^{\infty}\{p(y)X_j(y) - r(y)Y_j(y)\}e^{-1\mu y}\,(y-2x)dy$$

The first integral is convergent and the last two integrals tends to zero as $x \to \infty$, therefore

$$X_j(x) = e^{i\mu x}\{C_{j1}(\lambda) + 0(1)\}, \qquad (j = 1,2) \qquad (25)$$

Where

$$C_{j1}(\lambda) = 1 + \frac{1}{21\mu}\int_0^{\infty} e^{-1\mu y}\{p(y)K_j(y) - r(y)Y_j(y)\}\,dy \qquad (j = 1,2) \qquad (26)$$

Similarly

$$T_j(x) = e^{i\mu y}\{C_{j2}(\lambda) + 0(1)\}, \qquad (j = 1, 2) \qquad (27)$$

Where

$$C_{j2}(\lambda) = 1 + \frac{1}{2i\mu}\int_0^{\infty} e^{-i\mu y}\{q(y)Y_j(y) - r(y)X_j(y)\}\,dy \qquad (j = 1,2) \qquad (28)$$

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Cite this article as :

Rajeev Ranjan, "Eigenfunction Expansion and Spectral Theorem", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 11 Issue 1, pp. 280-285, January-February 2024. Available at doi : https://doi.org/10.32628/IJSRST52310634

Journal URL : https://ijsrst.com/IJSRST52310634