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Study of Some Contribution of Operator Algebra

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ABSTRACT

In this present paper, we studied about some classical applications of operator algebras in mathematics and mathematical physics. [1-2].

Keywords: Operator Algebras, Topology, Compact Operators, Vector-Space, Hilbert Spaces, Homomorphic, Functional Analysis.

I. INTRODUCTION

The purpose of this article is to give a brief and informal overview on C*- and von Neumann algebras. We will also mention some of the classical results in the theory of operator algebras that have been crucial for the development of several areas in mathematics and mathematical physics. We have also included a few exercises to motivate further thoughts on the subjects treated. In this section we present three different ways one may look at operator algebras.

Operator algebras as non-commutative spaces:

There are structure theorems stated in [1] saying that, essentially, the prototypes mentioned

$$(C0(X), \| \cdot \|) \quad \text{and} \quad L^\infty(Z, d\mu)$$

are the only possible commutative examples of C*- and von Neumann algebras, respectively. In the context of commutative C*-algebras it is also possible to recapture the topological space X from the algebraic structure of the set of continuous functions on X decaying at infinity. It is therefore rea- sonable to think of non-commutative C*-algebras as the non-commutative counterpart of topological spaces. In the same way non-commutative von Neumann algebras can be associated with non-commutative measure spaces. The correspondence

space ↔ algebraic structure

opens, in the non-commutative setting, a wide and difficult field of current research that includes advanced topics like non-commutative geometry, non-commutative L^p-spaces or quantum groups. [2-5].

Operator algebras as a natural universe for spectral theory:

In the present subsection we will argue that operator algebras are a natural universe for studying the properties of a single operator. The following proposition shows that the fundamental constituents in which one may decompose a single operator are contained in the corresponding von Neumann algebra. In other words, von Neumann algebras are stable under natural operations performed with its elements.

Proposition 1.1:

Let $M \subset L(H)$ be a von Neumann algebra and $M \in M$

(i) If $M = V\,|M\,|$ is the polar decomposition, then $V \in M \ni |M\,|$. (Recall that $|M\,| := (M^*M)^{\frac{1}{2}}$ is a positive operator and that V is a partial isometry satisfying $\ker V = \ker M$).

(ii) If $M = M^*$ and $M = \int \lambda dE_M(\lambda)$ is the corresponding spectral de-composition of the self-adjoint operator, then for the set of spectral projections we have

$$\{E_M(B) \mid B \subset \mathbb{R}\ ,\ \text{Borel}\,\} \subset \mathcal{M}$$

(iii) If $M = M^*$ and $f \in C([-\|M\|, \|M\|])$, then $f\,(M\,)$ is in any C*- algebra containing M. In particular, $f(M) \in \mathcal{M}$.

Proof. :

We sketch only a few ideas of the proof: to show that any operator is contained in the von Neumann algebra M, it is enough to verify that it commutes with all unitaries $U' \in M'$. To prove (i) note that for any $U' \in M'$ we have

$$V\,|M| = M = U'\,M\,(U')^* = (U'\,V\,(U')^*)(U'\,|M|\,(U')^*)\ .$$

From the uniqueness of the polar decomposition, we conclude that

$$(U'\,V\,(U')^*) = V \quad \text{and} \quad U'\,|M|\,(U')^* = |M| \quad \text{for all} \quad U' \in M'\ ,$$

hence $V, |M\,| \in M'' = M$. Item (ii) is shown similarly using the uniqueness of the spectral decomposition of self-adjoint operators. For (iii) take a sequence p_n of polynomials approximating f in the sup-norm. Then it follows that $p_n(M\,) \in M$ approximates in the operator-norm the operator f (M).

Hence

f (M) is in any C*-algebra containing M . Since any von Neumann algebra is also closed concerning the operator norm we conclude that $f(M) \in \mathcal{M}$.

The precedent proposition implies that any von Neumann algebra is generated as a norm closed subspace by the set of spectral projections corresponding to its self-adjoint elements.

II. SOME CLASSICAL RESULTS

In the present section, we recall some classical applications of operator algebras in mathematics and mathematical physics.

2.1 Operator algebras in functional analysis.

At the heart of the following results lies the structure theorem for commutative C*- and von Neumann algebras.

2.1.1 Spectral theorem. An immediate success of operator algebraic methods in functional analysis was the proof of the spectral theorem for bounded as well as unbounded normal operators on a Hilbert space. The spectral theorem is a generalization of the elementary result that any normal linear operator on C^n is unitary equivalent to a diagonal matrix. It can be stated in many ways. One of them says that any normal operator is equivalent to a multiplication operator. In applications, the spectral theorem is often stated in terms of the spectral resolution $E(\cdot)$ of a self-adjoint operator. (Recall that the orthogonal projections $\{E(\lambda)\}_\lambda$ satisfy the usual properties of monotonicity, right continuity and completeness.) For additional comments and results concerning the spectral theoremsee [7-9] and references therein.

Theorem 2.1. For any self-adjoint operator T on a complex Hilbert space H, there is a unique spectral resolution $E_T(\cdot)$ such that

$$T = \int_{sp(T)} \lambda dE_T(\lambda)$$

Here, sp (T) denotes the spectrum of the operator T and the right-hand in-integral is a Riemann-Stieltjes integral.

Finally, we mention a class of groups, where the previous decomposition results become particularly simple. A group G is of type I if all its unitary continuous representations U are of type I, i.e. each U is quasi-equivalent to some multiplicity-free representation. Compact or Abelian groups are examples of type I groups. If G is of type I, then the dual G (i.e. the set of all equivalence classes of continuous unitary irreducible representations of G) becomes a nice measure space ("smooth" in the terminology. In this case, one can take G as the measure space Z in the Mautner decomposition mentioned in the preceding item (ii).

Operator algebras in quantum physics:

The publication of the seminal books of Weyl, Wigner, and van der Waerden in the late twenties shows that quantum mechanics was using group theoretical methods almost from its birth. A nice summary of this circle of ideas can be found in [2]. Moreover, it is suggested by Ulam in that the spectral theorem and functional calculus are as fundamental to quantum mechanics, as infinitesimal calculus is for classical mechanics. Therefore, operator algebraic methods are indirectly present in quantum physics through the representation theory of groups and functional analysis. A direct application of operator algebraic methods in the first years of quantum theory was von Neumann's rigorous proof of the mathematical equivalence of the two main competing formalisms at that time: the wave mechanics of Schrödinger and the matrix mechanics of Born, Heisenberg, and Jordan; for a thorough historical account on the equivalence problem [13-15].

Remark 2.5. A brief historical introduction to the relation between the representation theory of groups and quantum mechanics is given. In this paper, the author also proposes K-theory for operator algebras as a new synthesis of these topics.

III. CONCLUSIONS

In particular, causality is expressed in this context in the following natural way: if Θ_1 and Θ_2 are space-like separated regions in Minkowski space, then $A(\Theta_1)$ commutes elementwise with $A(\Theta_2)$ for further details. Non-local aspects in quantum field theory like the notion of the vacuum, S-matrix etc. are related to the states. Local quantum physics complements other modern developments in relativistic quantum field theory and is particularly powerful in the analysis of structural questions as well as for the rigorous treatment of models. Algebraic quantum field theory has been very successfully applied in super selection theory, the theory that studies three characteristic aspects of elementary particle physics: composition of charges, classification of statistics and charge conjugation. For applications of Modular Theory to quantum field theory and references.

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