International Journal of Scientific Research in Science and Technology
Available online at : www.ijsrst.com
OPEN ACCESS
Print ISSN: 2395-6011 | Online ISSN: 2395-602X

A Study on W_3 – Curvature Tensor in Kenmotsu Manifolds

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ABSTRACT

In this paper we study some curvature properties of Kenmotsu manifolds satisfying the conditions W_3-Ricci pseudo-symmetric condition, $W_3 \cdot Q = 0$.

Keywords : Curvature Tensor, Kenmotsu Manifold, Einstein.

1. Introduction

The notion of Kenmotsu manifolds was defined by Kenmotsu [12]. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant. For example it is hyperbolic space (-1). Kenmotsu manifolds have been extensively studied by many authors like ([1], [2], [7], [15], [16]). Kenmotsu manifolds are a class of manifolds in differential geometry that are specifically designed to have a rich structure, particularly in the context of pseudo-Riemannian geometry. They are named after Kenmotsu, who first introduced them.

The concept of Ricci pseudo-symmetric manifold was introduced by Deszcz ([5], [6]). A geometrical interpretation of Ricci pseudo-symmetric manifolds in the Riemannian case is given in [11]. A Riemannian manifold (M, g) is called Ricci pseudosymmetric ([5], [6]) if the tensor $R \cdot S$ and the Tachibana tensor $Q(g, S)$ are linearly dependent, where

(1.1) $(R(X,Y) \cdot S)(Z,U) = -S(R(X,Y)Z,U) - S(Z,R(X,Y)U),$

(1.2) $Q(g,S)(Z,U;X,Y) = -S\left((X \wedge_g Y)Z,U\right) - S(Z,(X \wedge_g Y)U)$

and

(1.3) $(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y$

for all vector fields X, Y, Z, U of M, R denotes the curvature tensor of M. Then Kenmotsu manifold is Ricci pseudo-symmetric if and only if

(1.4) $(R(X,Y) \cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y)$

holds on $U_S = \left\{x \in M : S \neq \frac{r}{(2n+1)} g \ at \ x\right\}$, where L_S is some function on U_S. If $R \cdot S = 0$, then M is called Ricci semi-symmetric. Every Ricci semi-symmetric manifold is Ricci pseudo-symmetric but the converse is not true [6]. In this connection it is mentioned that in ([4], [9], [10], [14]) U.C. De, A.A. Shaikh, S.K. Hui et.al. studied Ricci pseudo-symmetric generalized quasi-Einstein manifolds.

Motivated by all these work in this paper we study some curvature properties of Kenmotsu manifold.

2. Preliminaries:

A $(2n+1)$- dimensional differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η on M satisfying [3]

(2.1) $\phi^2 X = -X + \eta(X)\xi, \ \ \phi\xi = 0,$
$\eta(\xi) = 1, \quad \eta \cdot \phi = 0.$

If g is a Riemannian metric with almost contact structure that is,

(2.2) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$
$\eta(X) = g(X, \xi).$

Then M is called an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) and denoted by (M, ϕ, ξ, η, g).

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies, $d\eta(X, Y) = g(X, \phi Y)$. Then (M, ϕ, ξ, η, g) is said to be a contact metric manifold.

A Kenmotsu manifold is a specific type of almost contact metric manifold. These manifolds have a contact structure and an associated pseudo-Riemannian metric. More formally, an almost contact metric manifold (M, ϕ, ξ, η, g) is said to be Kenmotsu manifold [12] if

(2.3) $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$

holds, where ∇ denotes the covariant differentiation. From (2.3), we get

(2.4) $\nabla_X \xi = X - \eta(X)\xi.$

In a Kenmotsu manifold M the following relations holds:

(2.5) $R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$
(2.6) $R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$
(2.7) $S(X, \xi) = -2n\, \eta(X),$
(2.8) $Q\xi = -2n\, \xi,$

for any vector fields X, Y. Where R is the Riemannian curvature tensor, Q is the Ricci operator i.e. $g(QX, Y) = S(X, Y)$ and S is the Ricci tensor of the manifold M.

3. Kenmotsu manifold satisfying W_3- Ricci pseudo-symmetric condition:

In this section, we study W_3 - Ricci pseudo-symmetric Kenmotsu manifold. In [13], Pokhariyal introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance and defined as

(3.1) $W_3(X, Y)Z = R(X, Y)Z + \frac{1}{2n}[g(Y, Z)QX - S(X, Z)Y]$

where R, S and Q are the curvature tensor, Ricci tensor and Ricci operator of the manifold respectively. From (3.1), we obtain the following:

(3.2) $W_3(\xi, Y)Z = 2k[\eta(Z)Y - g(Y, Z)\xi],$
(3.3) $W_3(X, Y)\xi = \eta(X)Y - \eta(Y)X + \frac{1}{2n}[\eta(Y)QX + 2n\,\eta(X)Y],$

for all vector fields X, Y, Z on M.

Kenmotsu manifold is said to be W_3- Ricci pseudo-symmetric if its W_3 - curvature tensor satisfies

(3.4) $(W_3(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y)$

holds on $U_S = \left\{x \in M : S \neq \frac{r}{(2n+1)} g \ at \ x\right\}$, where L_S is some function on U_S.

From (3.4), we get

(3.5) $S(W_3(X, Y)Z, U) + S(Z, W_3(X, Y)U)$
$= L_S[g(Y, Z)S(X, U) - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z)].$

Putting $Z = \xi$ in (3.5) and by virtue of (2.7), (3.3), we obtain

(3.6) $\eta(X)S(Y, U) - \eta(Y)S(X, U) + \frac{1}{2n}[\eta(Y)S(QX, U) + 2n\eta(X)S(Y, U)]$
$-2n[g(X, U)\eta(Y) - g(Y, U)\eta(X) + \frac{1}{2n}\{-2ng(Y, U)\eta(X) - \eta(Y)S(X, U)\}]$
$= L_S[\eta(Y)S(X, U) - \eta(X)S(Y, U) - 2ng(Y, U)\eta(X) + 2ng(X, U)\eta(Y)].$

Putting $Y = \xi$ in (3.6) with the help of (2.7) and on simplification, we get

(3.7) $S(QX, U) = 2nL_S S(X, U) + 4n^2(L_S + 1)g(X, U)$

which implies,

(3.8) $S^2(X, U) = 2nL_S S(X, U) + 4n^2(L_S + 1)g(X, U)$.

Hence, we state the following:

Theorem 3.1. If Kenmotsu manifold satisfying W_3-Ricci pseudo-symmetric, then the square of the Ricci tensor S^2 is equal to the linear combination of $2nL_S$ times of the Ricci tensor S and $4n^2(L_S + 1)$ times of the metric tensor g.

4. Kenmotsu manifold satisfying $W_3 \cdot Q = 0$:

In this section, we study Kenmotsu manifold satisfying $W_3 \cdot Q = 0$. Then, we have

(4.1) $W_3(X, Y)QZ - Q(W_3(X, Y)Z) = 0$.

Putting $Y = \xi$ in (4.1), we obtain

(4.2) $W_3(X, \xi)QZ - Q(W_3(X, \xi)Z) = 0$.

By virtue of (3.1) in (4.2), we get

(4.3) $\frac{1}{2n}S(X, QZ)\xi = 2ng(Z, X)\xi - \frac{1}{2n}\eta(Z)Q^2X + 2n\eta(Z)X$.

Taking inner product with ξ in (4.3) and on simplification, we have

(4.4) $S(X, QZ) = 4n^2 g(Z, X)$,

which implies,

(4.5) $S^2(X, Z) = 4n^2 g(Z, X)$.

Hence, we state the following:

Theorem 4.1. If Kenmotsu manifold satisfying $W_3 \cdot Q = 0$, then the square of the Ricci tensor S^2 is equal to $4n^2$ times of the metric tensor g.

5. Conclusion:

The Ricci pseudo-symmetric condition in a Kenmotsu manifold links the curvature tensor and Ricci tensor in a structured way. This condition provides significant insights into the curvature properties of Kenmotsu manifolds, often leading to classifications in terms of Einstein manifolds or constant curvature.

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