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A Study on Projective Curvature Tensor in Trans-Sasakian Manifolds

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ABSTRACT

In this paper we show that trans-Sasakian manifolds satisfying the conditions $R(X, Y) \cdot S = 0$, $P(\xi, X) \cdot S = 0$ are Einstein manifold.

Keywords : Trans-Sasakian Manifolds, Projective Curvature Tensor, Einstein.
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1 INTRODUCTION

A class of almost contact metric manifolds known as trans-Sasakian manifolds was introduced by J.A.Oubina [6] in 1985. This class contains α-Sasakian, β-Kenmotsu and co-symplectic manifolds.

Trans-Sasakian manifolds are an important generalization of Sasakian, Kenmotsu and co-symplectic manifolds in differential geometry. They arise naturally in the study of contact geometry and Riemannian geometry. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M \times R$ belongs to the class W_4, a class of Hermitian manifolds which are closely related to a locally conformal Kahler manifolds. Trans-Sasakian manifolds were studied extensively by J.C. Marrero [5], C.S. Bagewadi and Venkatesha [1, 2], M.M. Tripathi [9] and others. Trans-Sasakian manifolds are an important generalization of Sasakian and cosymplectic manifolds in differential geometry. They arise naturally in the study of contact geometry and Riemannian geometry. Trans-Sasakian manifolds are used in theoretical physics, particularly in string theory and contact mechanics. They also appear in Hamiltonian dynamics, differential geometry, and sub-Riemannian geometry. They provide a unifying framework to study different geometric structures that arise naturally in complex geometry and topology.

In this paper, we study the trans-Sasakian manifolds satisfing the conditions $R(X, Y) \cdot S = 0$, $P(\xi, X) \cdot S = 0$ are Einstein.

2.Preliminaries

An n-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits a $(1, 1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfy for all vector fields X, Y on M

(2.1) $$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1,$$

(2.2) $$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi),$$

(2.3) $$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be trans-Sasakian manifold if $(M \times R, J, G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M \times \boldsymbol{R}$ defined by $J\left(Z, f\frac{d}{dt}\right) = \left(\varphi Z - f\xi, \eta(Z)\frac{d}{dt}\right)$ for any vector field Z on M and smooth function f on $M \times \boldsymbol{R}$ and G is the product metric on $M \times R$. This may be stated by the condition

(2.4) $$(\nabla_X \varphi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\},$$

where α, β are smooth functions on M and such a structure is said to be the trans-Sasakian structure of type (α, β). From (2.4) it follows that

(2.5) $$\nabla_X \xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\}.$$

Note:

(1) If we consider α and β are smooth functions in equation (2.4) and $\alpha \neq 0$, $\beta = 0$ then the trans-Sasakian manifolds of type (α, β) reduces as α-Sasakian manifolds. Similarly, if α and β are scalars and $\alpha = 1$, $\beta = 0$ then the trans-Sasakian manifolds reduces as Sasakian manifolds.

(2) If we consider α and β are smooth functions in equation (2.4) and $\alpha = 0$, $\beta \neq 0$ then the trans-Sasakian manifolds of type (α, β) reduces as β-Kenmotsu manifolds. Similarly, if α and β are scalars and $\alpha = 0$, $\beta = 1$ then the trans-Sasakian manifolds reduces as Kenmotsu manifolds.

In a trans-Sasakian manifold $M(\varphi, \xi, \eta, g)$ the following relations hold:

$$R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2 Y$$
(2.6) $$+ 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2 X,$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - 2\alpha\beta[g(\varphi X, Z)\eta(Y) - g(\varphi Y, Z)\eta(X)]$$

$$- (Y\alpha)g(\varphi X, Z) - (X\beta)\{g(Y, Z) - \eta(Y)\eta(Z)\} + (X\alpha)g(\varphi Y, Z)$$
(2.7) $$+ (Y\beta)\{g(X, Z) - \eta(X)\eta(Z)\},$$

(2.8) $$R(\xi, X)\xi = (\alpha^2 - \beta^2 - (\xi\beta))[\eta(X)\xi - X],$$

(2.9) $$S(X, \xi) = [(n-1)(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\varphi X)\alpha) - (n-2)(X\beta),$$

(2.10) $$S(\xi, \xi) = [(n-1)(\alpha^2 - \beta^2) - (\xi\beta)],$$

(2.11) $$\xi\alpha + 2\alpha\beta = 0.$$

where R is the curvature tensor of type $(1, 3)$ and Q is the symmetric endomorphism of the tangent space at each point of the manifolds corresponding to the Ricci tensor S, that is, $g(QX, Y) = S(X, Y)$ for any vector fields X, Y on M.

Lemma 2.1. *In a trans-Sasakian manifold of type* (α, β), *if*

(2.15) $$\varphi(grad\alpha) = (n-2)(grad\beta),$$

then we have

(2.16) $$\xi\beta = 0.$$

Thus the directional derivative of β *with respect to characteristic vector field* ξ *is zero.*

The Weyl projective tensor P on Trans-Sasakian manifold $M$of dimensional n is defined by

(2.18) $$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[S(Y, Z)X - S(X, Z)Y],$$

for any vector fields X, Y, Z where R is the curvature tensor and r is the scalar curvature.

Trans-Sasakian manifolds satisfying $R(X, Y) \cdot S = 0$

Definition 3.1. *An n-dimensional trans-Sasakian manifold M is said to be Ricci semi-symmetric if*

(3.1) $$R(X, Y) \cdot S = 0,$$

for any vecotr fields X, Y where R is the curvature tensor and S is the Ricci tensor.

Theorem 3.1. *Let M be an n-dimensional trans-Sasakian manifold. Then M is Ricci-semi-symmetric if and only if an Einstein manifold.*

Proof. We know that every Einstein manifold is Ricci-semi-symmetric but the converse is not true in general. Here, we prove that in a trans-Sasakian manifolds $R(X, Y) \cdot S = 0$ implies that the manifold is an Einstein manifold.

(3.2) $$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0,$$

putting $X = \xi$ in equation (3.2), we have

(3.3) $$S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0.$$

By using (2.6) in (3.3), we obtain

$$\begin{aligned}
&(\alpha^2 - \beta^2)[g(Y, U)S(\xi, V) - \eta(U)S(Y, V) + g(Y, V)S(U, \xi) - \eta(V)S(U, Y)] \\
&+2\alpha\beta[g(\varphi U, Y)S(\xi, V) + \eta(U)S(\varphi Y, V) + g(\varphi V, Y)S(U, \xi) + \eta(V)S(U, \varphi Y)] \\
&+(U\alpha)S(\varphi Y, V) + g(\varphi U, Y)S(grad\alpha, V) + (U\beta)[S(Y, V) - \eta(Y)S(\xi, V)] \\
&-g(\varphi U, \varphi Y)S(grad\beta, V) + (V\alpha)S(U, \varphi Y) + g(\varphi V, Y)S(U, grad\alpha) \\
&+(V\beta)[S(U, Y) - \eta(Y)S(U, \xi)] - g(\varphi V, \varphi Y)S(U, grad\beta) = 0.
\end{aligned} \tag{3.4}$$

By putting $U = \xi$ in (3.4) and by using (2.9), (2.10), (2.11) and (2.16), we obtain

(3.5) $$S(Y, V) = (n-1)(\alpha^2 - \beta^2)g(Y, V).$$

Therefore, M is Einstein manifold. This completes the proof of the theorem

Trans-Sasakian manifolds satisfying $P(\xi, X) \cdot S = 0$

In this section, we consider $P(\xi, X) \cdot S = 0$ and prove the following theorem:

Theorem 5.3. *Let M be an n-dimensional trans-Sasakian manifold. If M satisfies the condition*

(4.1) $$P(\xi, X) \cdot S = 0,$$

then M is Einstein manifold and has scalar curvature $r = n(n-1)(\alpha^2 - \beta^2)$.

Proof. Since $P(\xi, X) \cdot S = 0$, we have

(4.2) $$P(\xi, X) \cdot S(Y, \xi) = 0.$$

This implies that

(4.3) $$S(P(\xi, X)Y, \xi) + S(Y, P(\xi, X)\xi) = 0.$$

In view of (2.19) in (4.3), we have

$$S((\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(Y)X] + 2\alpha\beta[g(\varphi Y, X)\xi + \eta(Y)\varphi X] + (Y\alpha)\varphi X + g(\varphi Y, X)(grad\alpha)$$

$$-g(\varphi Y, \varphi X)(grad\beta) + (Y\beta)[X - \eta(X)\xi] - \frac{1}{(n-1)}[S(X, Y)\xi - (n-1)(\alpha^2 - \beta^2)\eta(Y)X], \xi)$$

(4.4) $$+S(Y, (\alpha^2 - \beta^2)[\eta(X)\xi - X] - \frac{1}{(n-1)}[(n-1)(\alpha^2 - \beta^2)\{\eta(X)\xi - X\}]) = 0.$$

The above equation implies that

$$(\alpha^2 - \beta^2)[g(X, Y)S(\xi, \xi) - \eta(Y)S(X, \xi)] + 2\alpha\beta[g(\varphi Y, X)S(\xi, \xi) + \eta(Y)S(\varphi X, \xi)]$$
$$+(Y\alpha)S(\varphi X, \xi) + g(\varphi Y, X)S(grad\alpha, \xi) - g(\varphi Y, \varphi X)S(grad\beta, \xi) + (Y\beta)[S(X, \xi)$$

(4.5) $$-\eta(X)S(\xi, \xi)] - \frac{1}{(n-1)}[S(X, Y)S(\xi, \xi) - (n-1)(\alpha^2 - \beta^2)\eta(Y)S(X, \xi)] = 0.$$

By using (2.9), (2.10), (2.11), (2.16) and (2.17) in (4.5), we get

(4.6) $$(n-1)(\alpha^2 - \beta^2)^2 g(X, Y) - (\alpha^2 - \beta^2)S(X, Y) = 0.$$

This implies that

(4.7) $$S(X, Y) = (n-1)(\alpha^2 - \beta^2)g(X, Y).$$

On contracting (4.7), we have

(4.8) $$r = n(n-1)(\alpha^2 - \beta^2).$$

Therefore, M is an Einstein manifold with the scalar curvature $r = n(n-1)(\alpha^2 - \beta^2)$.

CONCLUSION

In a trans-Sasakian manifold if $R(X, Y) \cdot S = 0$and $P(\xi, X) \cdot S = 0$ then the manifold is Einstein manifold. Trans-Sasakian manifolds serve as a bridge between Sasakian, Kenmotsu and cosymplectic geometries, making them a rich area of study in modern differential geometry. Researchers continue to explore their curvature properties, classification, and applications in various fields of mathematics and physics. The projective curvature tensor is particularly useful in trans-Sasakian geometry, Einstein manifolds, and conformal geometry. Understanding its properties allows for deeper insights into the geometric and physical interpretations of various manifolds.

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