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Some results on Trans-Sasakian Manifolds

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ABSTRACT

In this paper we show that trans-Sasakian manifolds satisfying the conditions $R(X, Y) \cdot S = 0$, $\tilde{C}(\xi, X) \cdot S = 0$ are Einstein.

Keywords : Trans-Sasakian Manifolds, Concircular Curvature Tensor, Einstein.

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I. INTRODUCTION

A class of almost contact metric manifolds known as trans-Sasakian manifolds was introduced by J.A.Oubina [6] in 1985. This class contains α-Sasakian, β-Kenmotsu and co-symplectic manifolds.

Trans-Sasakian manifolds are an important generalization of Sasakian, Kenmotsu and co-symplectic manifolds in differential geometry. They arise naturally in the study of contact geometry and Riemannian geometry. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M \times R$ belongs to the class W_4, a class of Hermitian manifolds which are closely related to a locally conformal Kahler manifolds. Trans-Sasakian manifolds were studied extensively by J.C. Marrero [5], C.S. Bagewadi and Venkatesha [1, 2], M.M. Tripathi [9] and others. Trans-Sasakian manifolds are an important generalization of Sasakian and cosymplectic manifolds in differential geometry. They arise naturally in the study of contact geometry and Riemannian geometry. Trans-Sasakian manifolds are used in theoretical physics, particularly in string theory and contact mechanics. They also appear in Hamiltonian dynamics, differential geometry, and sub-Riemannian geometry. They provide a unifying framework to study different geometric structures that arise naturally in complex geometry and topology.

In this paper, we study the trans-Sasakian manifolds satisfying the conditions $R(X, Y) \cdot S = 0$ *and* $\tilde{C}(\xi, X) \cdot S = 0$ are Einstein where $\tilde{C}$ *is a* concircular curvature tensor.

Preliminaries

An n-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits a (1, 1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfy

(2.1) $$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1,$$

(2.2) $$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi),$$

(2.3) $$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M.

n almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be trans-Sasakian manifold if $(M \times R, J, G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M \times \mathbf{R}$ defined for any vector field Z on M and smooth function f on $M \times \mathbf{R}$ and G is the product metric on $M \times R$. This may be stated by the condition

(2.4) $$(\nabla_X \varphi) Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\},$$

where α, β are smooth functions on M and such a structure is said to be the trans-Sasakian structure of type (α, β). From (2.4) it follows that

(2.5) $$\nabla_X \xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\}.$$

Note:

(1) If we consider α and β are smooth functions in equation (2.4) and $\alpha \neq 0$, $\beta = 0$ then the trans-Sasakian manifolds of type (α, β) reduces as α-Sasakian manifolds. Similarly, if α and β are scalars and $\alpha = 1, \beta = 0$ then the trans-Sasakian manifolds reduces as Sasakian manifolds.

(2) If we consider α and β are smooth functions in equation (2.4) and $\alpha = 0$, $\beta \neq 0$ then the trans-Sasakian manifolds of type (α, β) reduces as β-Kenmotsu manifolds. Similarly, if α and β are scalars and $\alpha = 0$, $\beta = 1$ then the trans-Sasakian manifolds reduces as Kenmotsu manifolds.

In a trans-Sasakian manifold $M(\varphi, \xi, \eta, g)$ the following relations hold:

(2.6) $$R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2 Y + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2 X,$$

(2.7) $$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - 2\alpha\beta[g(\varphi X, Z)\eta(Y) - g(\varphi Y, Z)\eta(X)] - (Y\alpha)g(\varphi X, Z) - (X\beta)\{g(Y, Z) - \eta(Y)\eta(Z)\} + (X\alpha)g(\varphi Y, Z) + (Y\beta)\{g(X, Z) - \eta(X)\eta(Z)\},$$

(2.8) $$R(\xi, X)\xi = (\alpha^2 - \beta^2 - (\xi\beta))[\eta(X)\xi - X],$$

(2.9) $$S(X, \xi) = [(n-1)(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\varphi X)\alpha) - (n-2)(X\beta),$$

(2.10) $$S(\xi, \xi) = [(n-1)(\alpha^2 - \beta^2) - (\xi\beta)],$$

(2.11) $$\xi\alpha + 2\alpha\beta = 0.$$

where R is the curvature tensor of type (1, 3) and Q is the symmetric endomorphism of the tangent space at each point of the manifolds corresponding to the Ricci tensor S, that is, $g(QX, Y) = S(X, Y)$ for any vector fields X, Y on M.

Lemma 2.1. *In a trans-Sasakian manifold of type* (α, β), *if*

(2.15) $$\varphi(grad\alpha) = (n-2)(grad\beta),$$

then we have

(2.16) $$\xi\beta = 0.$$

Thus the directional derivative of β *with respect to characteristic vector field* ξ *is zero.*

The concircular curvature tensor $\tilde{C}$ on Trans-Sasakian manifold M of dimensional n is defined by

(2.18) $$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

for any vector fields X, Y, Z where R is the curvature tensor and r is the scalar curvature.

Trans-Sasakian manifolds satisfying $R(X, Y)\cdot S = 0$

Definition 3.1. *An n-dimensional trans-Sasakian manifold M is said to be Ricci semi-symmetric if*

(3.1) $$R(X, Y)\cdot S = 0,$$

for any vector fields X, Y where R is the curvature tensor and S is the Ricci tensor.

Theorem 3.1. *Let M be an n-dimensional trans-Sasakian manifold. Then M is Ricci-semi-symmetric if and only if an Einstein manifold.*

Proof. We know that every Einstein manifold is Ricci-semi-symmetric but the converse is not true in general. Here, we prove that in a trans-Sasakian manifolds $R(X, Y)\cdot S = 0$ implies that the manifold is an Einstein manifold.

(3.2) $$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0,$$

putting $X = \xi$ in equation (3.2), we have

(3.3) $$S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0.$$

By using (2.6) in (3.3), we obtain

$$(\alpha^2 - \beta^2)[g(Y, U)S(\xi, V) - \eta(U)S(Y, V) + g(Y, V)S(U, \xi) - \eta(V)S(U, Y)]$$
$$+2\alpha\beta[g(\varphi U, Y)S(\xi, V) + \eta(U)S(\varphi Y, V) + g(\varphi V, Y)S(U, \xi) + \eta(V)S(U, \varphi Y)]$$
$$+(U\alpha)S(\varphi Y, V) + g(\varphi U, Y)S(grad\alpha, V) + (U\beta)[S(Y, V) - \eta(Y)S(\xi, V)]$$
$$-g(\varphi U, \varphi Y)S(grad\beta, V) + (V\alpha)S(U, \varphi Y) + g(\varphi V, Y)S(U, grad\alpha)$$
(3.4) $$+(V\beta)[S(U, Y) - \eta(Y)S(U, \xi)] - g(\varphi V, \varphi Y)S(U, grad\beta) = 0.$$

By putting $U = \xi$ in (3.4) and by using (2.9), (2.10), (2.11) and (2.16), we obtain

(3.5) $$S(Y, V) = (n-1)(\alpha^2 - \beta^2)g(Y, V).$$

Therefore, M is Einstein manifold. This completes the proof of the theorem.

Trans-Sasakian manifolds satisfying $\tilde{C}(\xi, X)\cdot S = 0$ In this section we consider $\tilde{C}(\xi, X)\cdot S = 0$ and prove the following theorem:

Theorem 4.2. *Let M be an n-dimensional trans-Sasakian manifold. If M satisfies the condition*

(4.1) $$\tilde{C}(\xi, X)\cdot S = 0,$$

then M is Einstein manifold and has scalar curvature $r = n(n-1)(\alpha^2 - \beta^2)$.

Proof. Since $\tilde{C}(\xi, X)\cdot S = 0$, we have

(4.2) $$\tilde{C}(\xi, X) \cdot S(Y, \xi) = 0.$$

(4.3) $$S(\tilde{C}(\xi, X)Y, \xi) + S(Y, \tilde{C}(\xi, X)\xi) = 0.$$

In view of (2.18) in (4.3), we have

$$S((\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(Y)X] + 2\alpha\beta[g(\varphi Y, X)\xi + \eta(Y)\varphi X] + (Y\alpha)\varphi X + g(\varphi Y, X)(grad\alpha)$$

$$-g(\varphi Y, \varphi X)(grad\beta) + (Y\beta)[X - \eta(X)\xi] - \frac{r}{n(n-1)}[g(X, Y)\xi - \eta(Y)X], \xi)$$

(4.4) $$+S(Y, (\alpha^2 - \beta^2)[\eta(X)\xi - X] - \frac{r}{n(n-1)}[\eta(X)\xi - X]) = 0.$$

The above equation implies that

$$(\alpha^2 - \beta^2)[g(X, Y)S(\xi, \xi) - \eta(Y)S(X, \xi)] + 2\alpha\beta[g(\varphi Y, X)S(\xi, \xi) + \eta(Y)S(\varphi X, \xi)]$$

$$+(Y\alpha)S(\varphi X, \xi) + g(\varphi Y, X)S(grad\alpha, \xi) - g(\varphi Y, \varphi X)S(grad\beta, \xi) + (Y\beta)[S(X, \xi) - \eta(X)S(\xi, \xi)]$$

$$-\frac{r}{n(n-1)}[g(X, Y)S(\xi, \xi) - \eta(Y)S(X, \xi)] + (\alpha^2 - \beta^2)[\eta(X)S(Y, \xi) - S(Y, X)]$$

(4.5) $$-\frac{r}{n(n-1)}[\eta(X)S(Y, \xi) - S(Y, X)] = 0.$$

By using (2.9), (2.10), (2.11), (2.16) and (2.17) in (4.5), we get

(4.6) $$[(\alpha^2 - \beta^2) - \frac{r}{n(n-1)}][(n-1)(\alpha^2 - \beta^2)g(X, Y) - S(X, Y)] = 0.$$

This implies that

(4.7) $$S(X, Y) = (n-1)(\alpha^2 - \beta^2)g(X, Y).$$

On contracting (4.7), we have

(4.8) $$r = n(n-1)(\alpha^2 - \beta^2).$$

Therefore M is an Einstein manifold with the scalar curvature $r = n(n-1)(\alpha^2 - \beta^2)$.

Conclusion

In a trans-Sasakian manifold if $R(X, Y) \cdot S = 0$ *and* $\tilde{C}(\xi, X) \cdot S = 0$ then the manifold is Einstein manifold. Trans-Sasakian manifolds serve as a bridge between Sasakian, Kenmotsu and cosymplectic geometries, making them a rich area of study in modern differential geometry. Researchers continue to explore their curvature properties, classification, and applications in various fields of mathematics and physics. The concircular curvature tensor provides a refined way to measure the deviation of a manifold from constant curvature while preserving geodesic concircularity. It is particularly useful in trans-Sasakian geometry, Einstein manifolds, and conformal geometry. Understanding its properties allows for deeper insights into the geometric and physical interpretations of various manifolds.

REFERENCES

[1]. C.S.Bagewadi and Venkatesha, Some Curvature Tensors on a Trans-Sasakian Manifold, Turk. J. Math., 31, 111- 121, (2007).

[2]. C.S.Bagewadi and Venkatesha, Torseforming vector field in a 3-dimensional Trans-Sasakian Manifold, Differential Geometry-Dynamical Systems, 8, 23-28, (2006).

[3]. D.E.Blair and J.A.Oubina, Conformal and related changes of metric on the product of two almost contact metric manifolds, Publ. Matematiques, 34, 199-207, (1990).

[4]. U.C.De and M.M.Tripathi, Ricci tensor in 3-dimensional trans-Sasakian manifolds, Kyung- pook Math. J. 43, No.2, 247-255, (2003).

[5]. J.C.Marrero, The local structure of Trans-Sasakian manifolds, Annali di Matematica pura ed applicata. CLXII, 77-86, (1992).

[6]. J.A.Oubina, New classes of almost contact metric structures, Publ. Math. Debrecen, 32, 187-193, (1985).

[7]. A.A.Shaikh, K.K.Baishya and Eyasmin, On D-homothetic deformation of trans-Sasakian structure, Demonstr.

[8]. Math., XLI(1), 171-188, (2008).

[9]. A.Taleshian and N.Asghari, On LP-Sasakian manifolds satisfying certain conditions on the Concircular curvature tensor, Differential Geometry-Dynamical Systems, 12, 228-232, (2010).

[10]. M.M.Tripathi, Trans-Sasakian manifolds are generalised quasi-Sasakian, Nepali Math.Sci. Rep., 18, No.1-2, 11-14, (2001).