© 2017 IJSRST | Volume 3 | Issue 6 | Print ISSN: 2395-6011 | Online ISSN: 2395-602X
Themed Section: Science

Vertex Minimal Dominating Graph

Mr. Yashvanth N
Senior Scale Lecturer, Science Department, Government Polytechnic, Kushalnagar, Karnataka, India

Abstract

In this chapter, we study the vertex minimal dominating graph $M_vD(G)$ of a graph G and some of its properties. Also, we present some more basic results on vertex minimal dominating graph, in particular a characterization of $M_vD(G)$ which are complete, tree, Eulerian, Hamiltonian and planar. In addition we find the connectedness and diameter of $M_vD(G)$.

.

1. INTRODUCTION

The eccentricity $e(v)$ of v is defined by $e(v) = \max\{d(u.v)/u \in V(G)\}$.
The diameter diam (G) of G is defined by $diam(G) = \max\{e(u)/u \in V(G)\}$.

A graph is said to be embedded in a surface S when it is drawn on S so that no edges intersect geometrically except at a point to which they are both incident. A graph is called planar if it can be embedded in the plane; a planar graph has already been embedded in the plane.

A planar graph is called outerplanar if it can be embedded in the plane so that all its vertices lie on the same face. It is well-known that a graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - x$, where x is any edge of K_4.

The subdivision graph $S(G)$ of G is obtained by inserting a vertex in each edge.

The common minimal dominating graph $CD(G)$ of a graph G is the graph having the same vertex set as G with two vertices adjacent in $CD(G)$ if and only if there exists a minimal dominating set in G containing them. (see [4]).

The interesting graph valued function that is, the vertex minimal dominating graph of a graph is introduced by Kulli, Janakiram and Niranjan in [6].

The vertex minimal dominating graph $M_vD(G)$ of a graph G is a graph with $V(M_vD(G)) = V' = V \cup S$, where S is the collection of all minimal dominating sets of G with two vertices $u, v \in V'$ are adjacent if either they are adjacent in G or $v = D$ is a minimal dominating set of G containing u. We illustrate this concept through Fig 3.1.

Received : 02 July 2017 | Accepted : 24 July 2017 | July-August-2017 [(3) 6: 964-969]

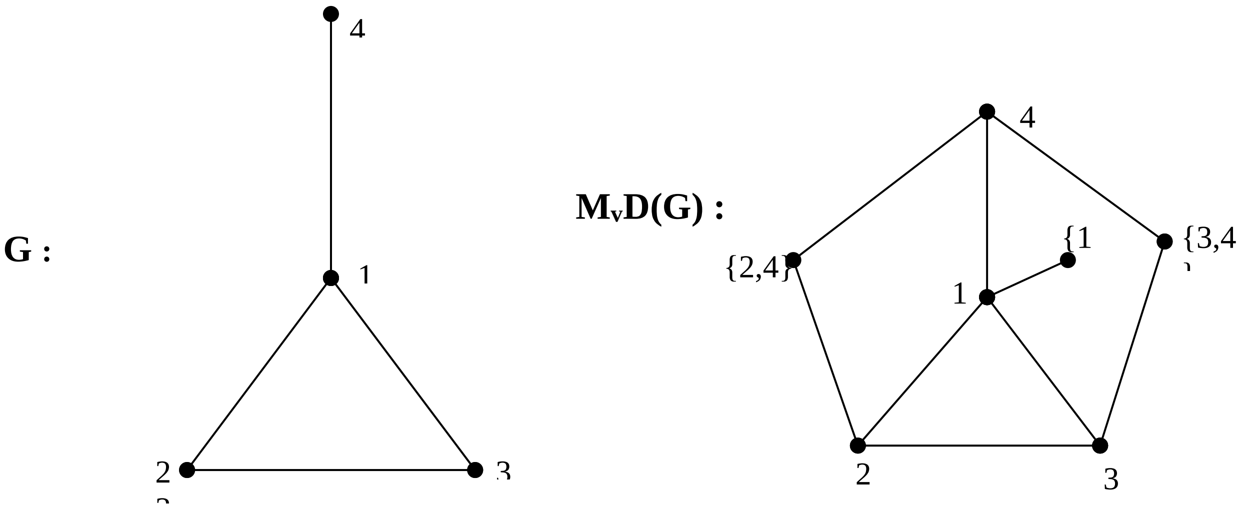

Figure: 3.1. A graph and its vertex minimal dominating graph

The following Theorems are useful to prove our next results.

THEOREM 3.A [3]. Every maximal independent set in a graph G is a minimal dominating set of G.

THEOREM 3.B [2]. A graph G is eulerian if and only if every vertex of G has even degree.

THEOREM 3.C [2]. A graph G is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

3.2. SOME BASIC PROPERTIES OF THE VERTEX MINIMAL DOMINATING GRAPH

First we present the characterizations of graph G for which $M_vD(G)$ is complete.

THEOREM 3.1. For any graph G, $M_vD(G)$ is connected. Further, it is complete if and only if $G = K_1$.

PROOF: Since for each $v \in V$ there exists a minimal dominating set containing v, every vertex in $M_vD(G)$ is not an isolated. Suppose $M_vD(G)$ is disconnected and G_1 and G_2 be two components of $M_vD(G)$. Then there exists two nonadjacent vertices $u, v \in V$ such that $u \in V_1' = V(G_1)$ and $v \in V_2' = V(G_2)$. This implies that there is no minimal dominating set in G containing u and v, which is a contradiction. Since there exists a maximal independent set containing u and v and by Theorem 3.A, every maximal independent set is a minimal dominating set. Hence, $M_vD(G)$ is connected. Now, we prove second part. Suppose $M_vD(G)$ is complete. Then G is complete and has exactly one minimal dominating set. This implies that $G = K_1$. Converse is obvious.
This completes the proof.

THEOREM 3.2. For any graph G,

$$\operatorname{diam}(M_vD(G)) \leq 3, \qquad \ldots (1)$$

where diam (G) is diameter of G.

PROOF: Suppose G has at least two vertices. Then $M_vD(G)$ has at least three vertices. Let $u, v \in V'$. We consider the following cases:

Case 1. Suppose $u, v \in V$. Then in $M_v D(G)$, $d(u,v) \le 2$.

Case 2. Suppose $u \in V$ and $v \notin V$. Then $v = D$ is a minimal dominating set of G. If $u \in D$, then in $M_v D(G)$, $d(u,v) = 1$. If $u \notin D$, then there exists a vertex $w \in D$ adjacent to u and hence in $M_v D(G)$

$$d(u,v) = d(u,w) + d(w,v) = 2.$$

Case 3. Suppose $u, v \notin V$. Then $u = D$ and $v = D'$ are two minimal dominating sets of G. If D and D' are disjoint set, then every vertex $w \in D$ is adjacent to some vertex $x \in D'$ and vice versa. This implies that in $M_v D(G)$

$$d(u,v) = d(u,w) + d(w,x) + d(x,v)$$

$$d(u,v) = 3.$$

If D and D' have a vertex in common, then in $M_v D(G)$

$$d(u,v) = d(u,w) + d(w,v)$$

$$d(u,v) = 2.$$

Thus, from Theorem 3.1, and above all the three cases (1) follows.
This completes the proof.
In the next result, they give bounds for order of $M_v D(G)$.

THEOREM 3.3. For any graph G,

$$p + d(G) \le p' \le \frac{p(p+1)}{2} \qquad \ldots (2)$$

Where $d(G)$ is the domatic number of G and p' denotes the number of vertices of $M_v D(G)$. Further, the lower bound is attained if and only if $G = K_p$ or $\overline{K_p}$ or $K_{1,p-1}$ and the upper bound is attained if and only if G is $(p-2)$ - regular.

PROOF: The lower bound follows from the fact that every graph has atleast $d(G)$ number of minimal dominating sets of G and upper bound follows from the fact that every vertex is in at most $(p-1)$ minimal dominating sets of G.

Suppose the lower bound is attained. Then every vertex is in exactly one minimal dominating set of G and hence, every minimal dominating set is independent. Further, for any two minimal dominating sets D and D' every vertex in D is adjacent to every vertex in D'. This implies the necessity. Sufficiency is straight forward. Suppose the upper bound is attained. Then each vertex is in exactly $(p-1)$ minimal dominating sets and hence G is $(p-2)$ - regular. Converse is obvious. This completes the proof.

In the next result, they give bounds for the size of $M_v D(G)$.

THEOREM 3.4. For any graph G,

(i) $\quad p + q \le q' \le p(p-1) + q \qquad \ldots (3)$

(ii) $\quad \{2q + p + (\gamma(G) \cdot d(G))\} / 2 \le q' \qquad \ldots (4)$

where q' denotes the number of edges of $M_v D(G)$. Further, the lower bound in (i) and (ii) are attained if and only if every vertex of G is in exactly one minimal dominating set of G and $G = K_p$ or $\overline{K_p}$ respectively and the upper bound in (i) attained if and only if G is $(p-2)$ - regular.

PROOF: First we prove (3). The lower bound follows from the fact that for every vertex $v \in V$ there is a minimal dominating set containing v.

Suppose the lower bound is attained. Then obviously each vertex is in exactly one minimal dominating set. Converse is obvious.

The proof for the upper bound is on the same lines of Theorem 3.3.

Now we prove (4).

$$2q' = \sum_{i=1}^{p^1} \deg(v_i) \qquad = \sum_{i=1}^{p} \deg(v_i) + \sum_{i=p+1}^{p^1} \deg(v_i)$$

$$2q' \geq \{2q + p + (\gamma(G) \cdot d(G))\}$$

$$q' \geq \{2q + p + (\gamma(G) \cdot d(G))\} \;/\; 2$$

This proves (4).

Suppose the bound is attained. On the contrary, if $G \neq K_p,\ \overline{K_p}$, then either there exists a vertex $v \in V$ which is in at least two minimal dominating sets of G or there exist two minimal dominating sets D and D' such that $|D| \neq |D'|$, which is a contradiction and hence, $G = K_p$ or $\overline{K_p}$. Converse is obvious. This completes the proof.

In the next result, a characterization is given for graphs G for which $M_vD(G)$ is a tree.

THEOREM 3.5. For any graph G, $M_vD(G)$ is a tree if and only if $G = \overline{K_p}$ or K_2.

PROOF: Suppose is a $M_vD(G)$ tree. Clearly, G has no cycle. On the contrary, if $G \neq \overline{K_p},\ K_2$, then we consider the following cases.

Case 1. If $\Delta(G) = p-1,\quad p \geq 3$, then G is a star and hence $M_vD(G)$ contains a cycle, a contradiction.

Case 2. If $\Delta(G) \leq p-2$, then there exists three vertices u, v and $w \in V$ such that u and v are adjacent and w is not adjacent to both u and v. This implies that in $M_vD(G)$, u and v are connected by at least two paths, once again a contradiction. Thus, from the above two cases necessity follows. Sufficiency is easy to see. This completes the proof.

COROLLARY 3.6. For any graph G,

$$\max\{p+q,\ \{2q + p + (\gamma(G) \cdot d(G))\} \;/\; 2\} \leq q' \qquad \ldots (5)$$

THEOREM 3.7. For any graph G,

$$\beta_0(M_vD(G)) = \max\{p' - p,\ \beta_0(G) + K\} \qquad \ldots (6)$$

where K is the maximum number of minimal dominating set in a vertex cover of G and $\beta_0(G)$ is the independence number of G.

PROOF: Let S' be a maximal independent set of vertices in $M_vD(G)$. Then $S' = S$ or $D_1 \cup S_1$, where S is the collection of all minimal dominating set of G, D_1 be the maximum independent set of vertices in G and S_1 be the collection of all minimal dominating sets of G in $V - D_1$ with $|S_1| = K$. This proves the theorem. This completes the proof.

COROLLARY 3.8. For any graph G.

(i) if G has no isolates, then

$$\beta_0(M_vD(G)) \geq \max\{D(G),\ \beta_0(G)+1\} \qquad \ldots (7)$$

(ii) for otherwise

$$\beta_0(M_vD(G)) \geq \beta_0(G). \qquad \ldots (8)$$

3.3. EULERIAN AND HAMILTONIAN PROPERTIES OF VERTEX MINIMAL DOMINATING GRAPH

They characterized vertex minimal dominating graphs which are eulerian.

THEOREM 3.9. For any graph G, $M_vD(G)$ is eulerian if and only if the following conditions are saticified:

(i) every minimal dominating set contains even number of vertices;

(ii) if $v \in V$ is a vertex of odd degree, then it is in odd number of minimal dominating sets, otherwise it is in even number of minimal dominating sets.

PROOF: Suppose $M_vD(G)$ is eulerian. On the contrary, if one of the given conditions say (i) is not satisfied, then there exist a minimal dominating set containing odd number of vertices and hence, $M_vD(G)$ has vertex of odd degree, a contradiction to Theorem 3.B. Hence, (i) holds. Similarly, we prove (ii). Suppose condition (ii) is not satisfied, then there exist a vertex v of odd degree and it is in even number of minimal dominating sets and hence, $M_vD(G)$ has vertex v of odd degree, a contradiction to Theorem 3.B.

Conversely, suppose the given conditions are satisfied. Then every vertex in $M_vD(G)$ has even degree and hence by the Theorem 3.B, $M_vD(G)$ is eulerian.

This completes the proof.

THEOREM 3.10. Let G be a $(p-3)$- regular graph and $\beta_0(G) = 2$. If every minimal dominating set of G is independent, then $M_vD(G)$ is hamiltonian.

PROOF: Since for each vertex $v \in V$ there exist two minimal dominating sets containing v and every minimal dominating set has exactly two vertices, $M_vD(G)$ is cycle and hence, it is hamiltonian.

This completes the proof.

They gave sufficient conditions on G for which the vertex minimal dominating graph of G is nonplanar.

THEOREM 3.11. Let $\langle B \rangle$ be a block in G satisfying the following conditions:

(i) $\langle B \rangle$ has three minimal dominating sets D_1, D_2 and D_3 such that each vertex $v \in S' = \bigcup_{i=1}^{3} Di$ is not adjacent to any vertex of $G - B$;

(ii) There exists a minimal dominating set for $G - B$ containing at least three vertices.

Then $M_vD(G)$ is non planar.

PROOF: Let D' be a minimal dominating set for $G - B$ containing atleast three vertices. Then for each D_i, $1 < i < 3$. $D' \cup D_1$ is a minimal dominating set for G. Thus, there exists three minimal dominating sets for G each of which contains the vertices of D'. This implies that $M_vD(G)$ contains $K_{3,\,3}$ as an induced subgraph and hence, by Theorem 3.C, it is nonplanar.

This completes the proof.

THEOREM 3.12. Let $\langle B \rangle$ be a block graph in G satisfying the following conditions:

(1) $\langle B \rangle$ has two minimal dominating sets D_1 and D_2 such that each vertex in $D_1 \cup D_2$ is not adjacent to any vertex of $G - B$.

(2) There exists a minimal dominating set for $G - B$ containing atleast three vertices.

Then $M_vD(G)$ is nonouterplanar.

THEOREM 3.13. If $\Gamma(G) = 2$ then,

$$S(CD(G)) \subseteq M_v D(G) \qquad \ldots (9)$$

where $\Gamma(G)$ is the upper domination number of G. Further, the equality holds if and only if $G = \overline{K_2}$.

PROOF: Let $u, v \in V$. If there exists a minimal dominating set $\{u, v\}$ in G, then in $CD(G)$, u and v are adjacent and in $M_v D(G)$ there exists a path of length two between u and v. This implies the proof of this theorem.

Now, we prove the second part.

Suppose the equality holds. Then $G = \overline{K_p}$. If $p \geq 3$, then there exists a minimal dominating set in G with at least three vertices, a contradiction. Hence, $G = \overline{K_2}$. Converse easy to prove.

This completes the proof.

References

[1]. Berge, C., Graphs and Hyper graphs. North-Holland, Amsterdam,(1973).

[2]. F. Harary, Graph Theory. Addision-Wesley, Reading, Mass, (1969).

[3]. Haynes, T.W., Hedetniemi, S.T and P.J. Slater, Fundamental of Domination in Graphs. Marcel Dekker, Inc, New York,(1998).

[4]. V.R.Kulli and B. Janakiram, The common minimal dominating graph. Indian,J. Pure Appl. Math.,27(1996), 193-196.

[5]. V.R. Kulli and B. Janakiram, The minimal dominating graph. Graph Theory Notes of New York, New York Academy of Sciences, 28(1995),12-15.

[6]. V.R. Kulli , B. Janakiram, and K.M. Niranjan, The vertex minimal dominating graph Acta Ciencia Indica, 28 (2002), 435-440.