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Dominating Set

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ABSTRACT :
In this chapter, we obtain the bounds on the number of edge and vertices edges, domatic number, domination number of the minimal dominating graph and vertex minimal dominating graph of a graph.
Keywords : Dominating Set, Minimal Dominating Set, Domination Number, Upper Domination Number, Domatic Number, Minimal Dominating Graph And Vertex Minimal Dominating Graph.

1. INTRODUCTION

The graph considered here are finite, undirected without loops or multiple edges. Any undefined term in this paper may be found in Harary[2].

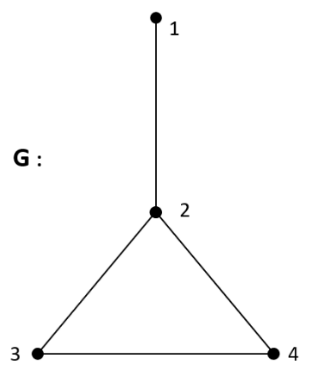
Let $G=(V,E)$ be a graph. A set $D \subseteq V$ is called a dominating set if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set if no proper subset $D' \subset D$ is a dominating set. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. The upper domination number $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set in G.

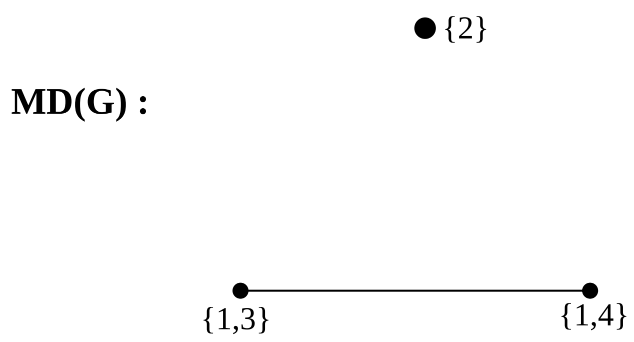
Domatic number $d(G)$ of a graph G to be the largest order of a partition of $V(G)$ into dominating set of G.

The minimal dominating graph $MD(G)$ of a graph G is the intersection graph defined on the family of all minimal dominating sets of vertices of G.

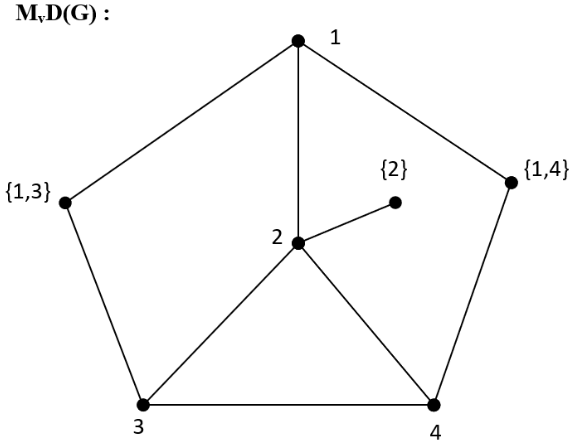
The vertex minimal dominating graph $M_vD(G)$ of a graph G is a graph with $V(M_vD(G)) = V' = V \cup S$, where S is the collection of all minimal dominating sets of G with two vertices $u, v \in V'$ are adjacent if either they are adjacent in G or $v = D$ is a minimal dominating set of G containing u.

In Fig.1, a graph G, its minimal dominating graph $MD(G)$ and vertex minimal dominating graph $M_vD(G)$ are shown.





$M_vD(G)$:



The following results are useful to prove our next results.

REMARK 1. The degree of the vertices of vertex minimal dominating graph $M_vD(G)$ is given by,

(i) $\deg_{M_vD(G)}(D_i) =$ cardinality of D_i in G

(ii) $\deg_{M_vD(G)}(v_j) = \deg_G(v_j) + t_j$

where $D_i, 1 \le i \le n$ denotes the minimal dominating sets of G and $t_j, 1 \le j \le p$ denotes the number of minimal dominating sets containing v_j in G.

REMARK 2. For any graph G,the set S= $\{S_1, S_2, \ldots\ldots, S_n\}$ is independent set of $M_vD(G)$

Where $S_i, 1 \le i \le n$ denotes the all minimal dominating sets of *G.*

THEOREM A [3]. For any graph G ,

$$\gamma(MD(G)) = P$$

if and only if every independent set of G is a dominating set.

THEOREM B [3]. For any graph G , $MD(G)$ is complete if an only if G contains an isolated vertex.

THEOREM C [4]. For any graph G , $M_vD(G)$ is tree if and only if $G = \overline{K_P}$ or K_2 .

THEOREM D [5]. If $\Gamma(G) \le 2$, then

$$S(\overline{G}) \subset D(G)$$

where $S(G)$ is the subdivision graph of G .

THEOREM E [4]. For any graph G ,

$$D(G) \subseteq M_vD(G)$$

Further, the equality holds if and only if $G = \overline{K_P}$.

THEOREM F[1].

(i) $d(K_n) = n$; $d(\overline{K}_n) = 1$

(ii) for any tree T with $p \ge 2$ vertices, $d(T) = 2$

RESULTS ON MINIMAL DOMINATING GRAPH

THEOREM 1. For any graph G ,

$$d(G) \le p' \le p(p-1) / 2$$

where p' denotes the number of vertices of $MD(G)$. Further the lower bound attained if an only if $G = K_p$ or $\overline{K_p}$ or $K_{1,\, p-1}$ and the upper bound is attained if and only if G is $(p-2)$ - regular.

PROOF: The lower bound follows from the fact that every graph has at least $d(G)$ number of minimal dominating sets of G and the upper bound follows from the fact that every vertex is in at most $(p-1)$ minimal dominating sets of G.

Suppose the lower bound is attained. Then every vertex is in exactly one minimal dominating set of G, and hence every minimal dominating set is independent. Further, for any two minimal dominating sets D and D' every vertex in D is adjacent to every vertex in D'. This implies the necessity.

Conversely, suppose $G = K_p$ or $\overline{K_p}$ or $K_{1,\, p-1}$. Then by Theorem F, $d(K_p) = p$ or $d(\overline{K}p) = 1$ or $d(K_{1,p-1}) = 2$ which implies that order of $MD(G)$ are p or one or two respectively.

Suppose the upper bound is attained. Then each vertex is in exactly $(p-1)$ minimal dominating sets and hence G is $(p-2)$ - regular.

Converse is obvious. This completes the proof.

THEOREM 2. For any graph G,

$$0 \le q' \le p(p-1)$$

where q' is the number of edges in $MD(G)$, further the lower bound attained if and only if $G = K_P$ or $\overline{K_P}$ or $K_{1,P-1}$ and the upper bound is attained if and only if G is $(p-2)$ - regular.

PROOF: Suppose the lower bound attains. Then $MD(G)$ is totally disconnected or K_1. Consequently $G = K_P$ or $\overline{K_P}$ or $K_{1,P-1}$.

Conversely, suppose $G = K_P$, then each vertex of G is a minimal dominating set of G. Hence $MD(G)$ is totally disconnected.

Suppose if $G = K_{1,\, p-1}$, then clearly, G has only two minimal dominating sets with no element in common. Hence $MD(G)$ is disconnected.

Suppose $G = \overline{K_P}$. Then $V(G)$ is the minimal dominating set of G. Hence $MD(G) = K_1$.

Suppose the upper bound is attained. Then each vertex of G is in exactly $(p-1)$ - minimal dominating sets and hence G is $(p-2)$ - regular.

Conversely, suppose G is $(p-2)$ - regular. Then clearly each vertex of G is in exactly $(p-1)$ - minimal dominating sets of G and in G we have p number vertices, which implies $MD(G)$ has $p(p-1)$ edges.

This completes the proof.

THEOREM 3. For any graph G,

$$\gamma(G) + \gamma(MD(G)) = p + 1$$

if and only if every independent set of G is a dominating set or $G = \overline{K_p}$.

PROOF: Suppose every independent set of G is dominating set. Then each $\{v\} \subseteq V$ is a minimal dominating set of G, this prove that $MD(G) = \overline{K_p}$. Hence $\gamma(G) + \gamma(MD(G)) = p + 1$ holds.Suppose

$G=\overline{K_p}$. Then $V(G)$ is a minimal dominating set of G . This implies $MD(G)=K_1$. Hence $\gamma(G)+\gamma(MD(G)) = p+1$ holds.

Conversely,suppose $\gamma(G)+\gamma(MD(G)) = p+1$ holds. On the contrary suppose $G \neq K_p$. Then there exist at least two non_adjecent vertices u and v in G .Clearly each vertex $w \in V(G)$ other than u and v form a minimal dominating set of G .Also the set $\{u,v\}$ form minimal dominating set of G .Consequently this gives $\gamma(G)=1$ and $\gamma(MD(G)) = p-1$,which is a contradiction. Therefore $G=K_p$.

Suppose $G \neq \overline{K_p}$ then there exist at least one non-trivial componet G_1 in G .In G we have two minimal dominating sets of order p-1 ,consequently this gives $\gamma(G) = p-1$ and $\gamma(MD(G)) = 1$, which is a contradiction .Therefore $G = \overline{K}_p$.

This completes the proof.

THEOREM 4. For any graph G ,

$$d(MD(G)) = |V(MD(G))|$$

if and only if G contains an isolated vertex.

PROOF: Suppose $d(MD(G)) = |V(MD(G))|$ holds. Then by Theorem F, $MD(G)$ is complete. And also by the Theorem B, $MD(G)$ is complete if and only if G contains an isolated vertex.

Conversely, suppose G contains an isolated vertex. Then by Theorem B, $MD(G)$ is complete and also by Theorem F,we have $d(MD(G)) = |V(MD(G))|$. This completes the proof.

THEOREM 5. For any graph G ,

$$\gamma(MD(G)) = 1$$

if and only if G contains an isolated vertex.

PROOF: Suppose $\gamma(MD(G)) = 1$. Then, $MD(G)$ is complete. And also by Theorem B, $MD(G)$ complete if and only if G contains an isolated vertex.Hence G contains isolated vertex.

Conversely, suppose G contains an isolated vertex, then by Theorem B, $MD(G)$ is complete which implice $\gamma(MD(G)) = 1$.

This completes the proof.

RESULTS ON VERTEX MINIMAL DOMINATING GRAPH

THEOREM 6. For any graph G , $M_vD(G)$ is bipartite if and only if $G=\overline{K_p}$ or $K_{1,P-1}$.

PROOF: Suppose $M_vD(G)$ is bipartite, then we have to prove that $G=\overline{K_p}$ or $K_{1,P-1}$. On the contrary if $G \neq \overline{K_p}$, then there exists a component G_1 of G which is not trivial. Then, clearly $M_vD(G)$ contains a cycle of length five, which is a contradiction. Hence $G=\overline{K_p}$. Suppose if $G \neq K_{1,\ P-1}$, then there exist a cycle in G . Since G is subgraph of $M_vD(G)$, this implies that $M_vD(G)$ contains a cycle of odd length (length three), which is again a contradiction. Hence $G = K_{1,\ P-1}$.

Conversely, suppose $G = \overline{K_p}$, then clearly by Theorem C, $M_v D(G)$ is tree this implies that $M_v D(G)$ is bipartite.

Suppose $G = K_{1,P-1}$, then there exist exactly two minimal dominating sets D and D' . D contains a vertex u of degree $p-1$ and D' contains the $V(G)-u$ vertices of degree one. Clearly by definition, $M_v D(G)$ we get the bipartite graph.

This completes the proof.

THEOREM 7. For any graph,

$$\kappa(M_v D(G)) = \min\left\{\min_{1\le i\le n}\{\deg_{M_v D(G)}(D_i)\},\ \min_{1\le j\le p}\{\deg_{M_v D(G)}(v_j)\}\right\}$$

PROOF: We consider the following cases:

Case 1. Let u be the vertex of $M_v D(G)$ which corresponds to the minimal dominating set of G . If it has the minimum degree among the other all vertices of $M_v D(G)$, then by deleting the vertices of $M_v D(G)$ which are adjacent to u , results in a disconnected graph. Thus,

$$\kappa(M_v D(G)) = \min_{1\le i\le n}\{\deg_{M_v D(G)}(D_i)\}$$

Case 2. Let w be the vertex of $M_v D(G)$ which corresponds to the vertex of G . If it has the minimum degree among all the other vertices of $M_v D(G)$. Then by deleting vertices of $M_v D(G)$ which are adjacent to w results in a disconnected graph. Thus,

$$\kappa(M_v D(G)) = \min_{1\le j\le p}\{\deg_{M_v D(G)}(v_j)\}$$

By combining the above two cases we get,

$$\kappa(M_v D(G)) = \min\left\{\min_{1\le i\le n}\{\deg_{M_v D(G)}(D_i)\},\ \min_{1\le j\le p}\{\deg_{M_v D(G)}(v_j)\}\right\}$$

This completes the proof.

THEOREM 8. For any graph G,

$$\lambda(M_v D(G)) = \min\left\{\min_{1\le i\le n}\left\{\deg_{M_v D(G)}(D_i)\right\},\ \min_{1\le j\le p}\left\{\deg_{M_v D(G)}(v_j)\right\}\right\}$$

PROOF: We consider the following cases:

Case 1. Let u be the vertex of $M_v D(G)$ which corresponds to the minimal dominating set of G. If it has the minimum degree among the other all vertices of $M_v D(G)$, then by deleting the edges in $M_v D(G)$ which are incident with u the resulting graph will be disconnected. Thus,

$$\lambda(M_v D(G)) = \min_{1\le i\le n}\left\{\deg_{M_v D(G)}(D_i)\right\}$$

Case 2. Let w be the vertex of $M_v D(G)$ which correspond to the vertex of G. If it has the minimum degree among the all other vertices of $M_v D(G)$, then by deleting the edges in $M_v D(G)$ which are incident with w. The resulting graph will be disconnected. Thus,

$$\lambda(M_v D(G)) = \min_{1\le j\le p}\left\{\deg_{M_v D(G)}(v_j)\right\}$$

By combining the above two cases we get,

$$\lambda(M_v D(G)) = \min\left\{\min_{1\le i\le n}\left\{\deg_{M_v D(G)}(D_i)\right\},\ \min_{1\le j\le p}\left\{\deg_{M_v D(G)}(v_j)\right\}\right\}$$

This completes the proof.

THEOREM 9. For any graph G,

$$\gamma(M_v D(G)) = p$$

if and only if $G = K_P$.

PROOF: Suppose $\gamma(M_v D(G)) = p$ holds. On the contrary, if $G \ne K_P$, then there exist at least two non-adjacent vertices u and v in G. Clearly each vertex $w \in V(G)$ other than u and v form a minimal dominating set of G. Also the set $\{u, v\}$ form a minimal dominating set of G. Consequently this gives $\gamma(M_v D(G)) = P-1$, which is a contradiction. Hence $G = K_P$.

Conversely, suppose $G = K_P$, then each $\{v\} \subseteq V(G)$ is a minimal dominating set of G. By the definition of $M_v D(G)$ each vertex is adjacent to exactly one minimal dominating set, which follows that $\gamma(M_v D(G)) = p$.

This completes the proof.

THEOREM 10. For any graph G,

$$d(M_v D(G)) = 2$$

if and only if $G = \overline{K_P}$ or K_2.

PROOF: Suppose $d(M_v D(G)) = 2$. Then by Theorem F, $M_v D(G)$ is a tree and also by Theorem C, we have $G = \overline{K_P}$ or K_2.

Conversely, suppose $G = \overline{K_P}$ or K_2. Then Theorem C, $M_vD(G)$ is a tree ,also by Theorem F, $d(M_vD(G)) = 2$.

This completes the proof.

THEOREM 11. If $\Gamma(G) = 2$, then

$$S(\overline{G}) \subseteq M_vD(G)$$

Further, the equality holds if and only if $G = \overline{K_2}$.

PROOF: Since by the Theorem D, $S(\overline{G}) \subset D(G)$. Also, from Theorem E, $D(G) \subseteq M_vD(G)$. Then $S(\overline{G}) \subseteq M_vD(G)$ holds.

Now, we have to prove second part.

Suppose the equality holds. On the contrary, if $G = \overline{K_P}$, for $p \geq 3$ then there exist a minimal dominating set in G with at least three vertices, a contradiction. Hence $G = \overline{K_2}$.

Conversely, suppose $G = \overline{K_2}$, then there exist a minimal dominating set D containing two vertices, say u and v of G. By definition of $M_vD(G)$, u and v are adjacent to D in $M_vD(G)$. Clearly which gives the path P_3. Also we know that $\overline{G} = K_2$ and $S(\overline{G}) = P_3$, therefore we have

$$S(\overline{G}) = M_vD(G)$$

This completes the proof.

THEOREM 12. For any graph G,

$$\chi(M_vD(G)) = \begin{cases} \chi(G)+1 & \text{If vertices of any minimal dominating set are colored with } \chi(G) \text{ colors.} \\ \chi(G) & \text{otherwise} \end{cases}$$

PROOF: Let G be a graph with $\chi(G) = K$, and D be the set of all minimal dominating sets of G. By Remark 2, D is independent . In the coloring of $M_vD(G)$, either we can make use of the colors which are used to color G, that is $\chi(M_vD(G)) = K = \chi(G)$.

Or, we should have to use one more new color. In particular , if the vertices of any minimal dominating set x of G have colored with K colors. Then we required one more new color to color x in $M_vD(G)$. Hence in this case we required $K+1$ colors to color $M_vD(G)$. Therefore,

$$\chi(M_vD(G)) = K+1$$

$$\Rightarrow \quad \chi(M_vD(G)) = \chi(G)+1$$

This completes the proof.

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