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# Certain Classes of Analytic Functions Involving Multiplier Transformations by using Q-Calculus Operators

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## ABSTRACT

In this paper we define the classes $T^{\lambda}(m,l,A,B)_q$ using Janowski class, multiplier transformations and q-calculus. The results investigated for these classes of functions include the co-efficient estimates, inclusion relations distortion bounds. Extreme points and many more properties.

**Keywords :** Univalent Functions, Janowski Class, Multiplier Transformation.
MSC 2010 Classifications: 30C45, 30C50.

## 1. Introduction

Let A denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

defined in the unit disc U = {z:|z|<1}.

Let T denote the subclass of A and U, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in T$ if it has a Taylor expression of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \tag{1.2}$$

Which is analytic in the open disc U

The q-shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(a,q)_n = \begin{cases} 1, & n = 0; \\ (1-\alpha)(1-\alpha q)...(1-\alpha q^{n-1}), & n \in \mathbb{N}. \end{cases} \quad (1.3)$$

And in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q_n)}{\Gamma_q(\alpha)}, (n>0), \quad (1.4)$$

Where the q-gamma functions [2,3] is defined by

$$\Gamma_q(x) = \frac{(q;q)_{\infty}(1-q)^{1-x}}{(q^x;q)_{\infty}} (0<q<1) \quad (1.5)$$

Note that, if |q|<1, the shifted factorial (1,3) remains meaningful for n = ∞ as a convergent infinity product

$$(\alpha;q)_{\infty} = \prod_{m=0}^{\infty}(1-\alpha q^m)$$

Now recall the following q-analogue definitions given by Gasper and Rahman[2]. The recurrence relation of q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), where [x]_q = \frac{1-q^x}{1-q} \quad (1.6)$$

and called q-analogue of x

Jacksons q-derivative and q-integral of a function f is defined as a subset of $\mathbb{C}$ are respectively, given by (see Gasper and Rahman [2])

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, (z \neq 0, q \neq 0), \tag{1.7}$$

$$\int_0^z f(t) d_q(t) = z(1-q) \sum_{m=0}^{\infty} q^m f(zq^m) \tag{1.8}$$

In view of the relation.

$$\lim_{q \to 1^-} \frac{(q^{\infty}; q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.9}$$

We observe that the q-shifted fractional (1,2) reduces to the familiar Pochhammer symbol $(\alpha)_n, where\ (\alpha)_n = \alpha(\alpha+1)...(\alpha+n+1)$.

For $-1 \leq A < B \leq 1$, let P (A,B)[4] denote the class of functions which are of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

Where $\omega$ is abounded analytic function satisfying the condition $\omega(0) = 0\ and\ |w(z)| \leq 1$

For $f \in A, m \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$, the operator [7] I(m,λ,l) f(z) is defined by

$$I(m,\lambda,l) f(z) = z - \sum_{k=2}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l+1} \right)^m a_k z^k \tag{1,10}$$

We say that a function $f \in T$ is in a class $T_q^{\lambda}(m,l,A,B)$ if

$$\frac{zDq\left[I\ m,\lambda,l\ \ f\ \ z\ \right]}{I\ m,\lambda l\ \ f\ \ z} = \frac{1+A\omega\ \ z}{1+B\omega\ \ z},\ m \in N_0\ , \qquad (1.11)$$

for $-1 \le A < B \le 1$, l, $\lambda > 0$ and for all $z \in U$.

Note that $T_q^0\ 1,0,2\alpha-1,1 = S_j^*\ \alpha$ introduced by Chatterjea [1] and $T_j^1\ 1,0,2\alpha-1,1 = C_j\ \alpha$ studied by Srivastava [6]. In particular we get the classes studied by Ravikumar [5]

## 2. Main Results

**Theorem 2.1** A function $f \in T$ is in the class $T_q^\lambda\ m,1,A,B$ if and only if

$$\sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m \left[\ k\ _q\ \ B+1\ -\ A+1\ \right] ak \le B-A$$

for $m \in N_0$, $-1 \le A < B \le 1$, $l$, $\lambda > 0$, and $z \in U$.

Proof: Since $f \in T_j^\lambda\ \ m,l,A,B$ , we have

$$\frac{z\,Dp\left[I\ m,\lambda,l\ \ f\ \ z\ \right]}{I\ m,\lambda l\ \ f\ \ z} = \frac{1+A\omega\ \ z}{1+B\omega\ \ z},\ m \in N_0\ , \qquad 2.1$$

$$\left\{Bz - \sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m a_k B z^k\ \ k\ _q - Az + \sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m a_k A z^k\right\}\omega\ \ z$$

$$= z - \sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m a_k z^k - z + \sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m a_k z^k\ \ k\ _q$$

*By schwarz's lemma, we get*

$$\left|\frac{\sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m a_k z^k\ \ k\ _q - 1}{B-A\ +\sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m\ A-B\ k\ _p\ \ a_k z^k}\right| \le 1$$

$$\sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m a_k z^k\left[k_p\ \ B+1\ -\ A+1\ \right] \le B-A$$

**Theorem 2.2.** The class $T_q^{\lambda}(m,l,A,B)$ is closed under convex linear combination.

Proof. Let $f(z), g(z) \in T_q^{\lambda}(m,l,A,B)$ and let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

For η such that 0 ≤ η ≤ 1 it suffices to show that the function defined by h (z) = (1–η) *f* (z) + ηg (z), z ∈ u belongs to $T_q^{\lambda}(m,l,A,B)$. Now $h(z) = z - \sum_{k=2}^{\infty}\left[(1-\eta)a_k + \eta b_k\right] z^k$.

Applying Theorem2.1, to $f(z), g(z) \in T_q^{\lambda}(m,l,A,B)$, We have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m \left[k_q(B+1)-(A+1)\right]\left[(1-\eta)a_k + \eta b_k\right] \\
&= (1-\eta)\sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m \left[k_q(B+1)-(A+1)\right] a_k \\
&+ \eta\sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m \left[k_q(B+1)-(A+1)\right]\left[(1-\eta) b_k\right. \\
&\leq (1-\eta)(B-A) + \eta(B-A) \\
&= (B-A)(1-\eta+\eta) = (B-A)
\end{aligned}$$

This implies that $h(z) \in T_q^{\lambda}(m,l,A,B)$.

**Theorem 2.3.** Let for i = 1,2,…..,k, $f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \in T_q^{\lambda}(m,l,A,B)$ and $0 < \beta_i < 1$ such that $\sum_{i=1}^{k} \beta_i = 1$, then the function F(z) defined by $F(z) = \sum_{i=1}^{k} \beta_i f_i(z)$ is also in $T_q^{\lambda}(m,l,A,B)$.

Proof. For each i ∈ {1, 2, 3, ……, k} we obtain

$$\sum_{k=2}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m \left[k_p(B+1)-(A+1)\right]\left|a_{k,i}\right| < (B-A)$$

$$Since\ F\ z\ =\sum_{i=1}^{k}\beta_i\left(z-\sum_{k=2}^{\infty}a_{k,i}z^k\right)$$

$$=\sum_{i=1}^{k}\beta_i z-\sum_{i=1}^{k}\left(\sum_{k=2}^{\infty}a_{k,i}z^k\right)$$

$$=z-\sum_{i=l}^{k}\left(\sum_{k-2}^{\infty}a_{k,i}\right)z^k.$$

$$\sum_{k=2}^{\infty}\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m\left[\ k\ _p\ \ B+1\ -\ A+1\ \right]\left[\sum_{i=1}^{k}\beta_i a_{k,i}\right]$$

$$=\sum_{i=1}^{k}\beta_i\left(\sum_{k=2}^{\infty}\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m\left[\left[k_a\right]\ B+1\ -\ A+1\ \right]a_{k,i}\right]$$

$$<\sum_{i=l}^{k}\beta_i\ \ B-A\ <\ B-A$$

Therefore $F\ z\ \in T_q^{\lambda}\ \ m,l,A,B\ .$

**Theorem 2.4.** Let $f\ \ z\ \in T_q^{\lambda}\ \ m,l,A,B$ Define $f_1(z) = z$ and

$$f_k\ \ z\ =z-\frac{B-A}{\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m\left[\ k\ _q\ \ B+1\ -\ A+1\ \right]}z^k, k\geq 2,$$

For some $-1\leq A < B \leq 1$, $m \in N_0$, $l, \lambda > 0$ and $z \in u$. Then $f\in T_q^{\lambda}\ \ m,l,A,B$ if and only if $f$ can be expressed as

$$f\ z\ =\sum_{k=1}^{\infty}\mu_k f_k\ \ z\ ,\ where\ \mu_k\geq 0\ and\ \sum_{k=1}^{\infty}\mu_k=1.$$

Proof. If $f\ \ z\ =\sum_{k=1}^{\infty}\mu_k f_k\ \ z\ ,\ with\ \sum_{k=1}^{\infty}\mu_k=1,\mu_k\geq 0,\ then$

$$\sum_{k=2}^{\infty}\frac{\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m\left[\ k\ _q\ \ B+1\ -\ A+1\ \right]}{\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m\left[\ k\ _q\ \ B+1\ -\ A+1\ \right]}\mu_k\ \ B-A$$

$$=\sum_{k=2}^{\infty}\mu_k = 1\; B-A \;=\; 1-\mu_j \quad B-A$$

$\le$ (B – A)

Hence $f\ z \in T_q^{\lambda}\ m, l, A, B$

Conversely, let $f\ z = z - \sum_{k=2}^{\infty} a_k z^k \in T_q^{\lambda}\ m,l,A,B$ ,

$$define\ \ \mu_k = \frac{\left(\frac{\lambda\ k-1\ +l+1}{l+1}\right)^m \left[\, k\ _q\ \ B+1\ -\ A+1\,\right]\left|a_k\right|}{B-A}, k \ge 2$$

And define $\mu_k = 1-\sum_{k=2}^{\infty}\mu_k$. From Theorem 2.1, $\sum_{k=2}^{\infty}\mu_k \le 1$ and hence $\mu_1 \ge\ 0$.

Since $\mu_k f_k\ (z) = \mu_k\ f(z) + a_k z^k$,

$$\sum_{k=2}^{\infty}\mu_k f_k\ \ z\ = z - \sum_{k=2}^{\infty} a_k z^k = f\ \ z\ .$$

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