© 2018 IJSRST | Volume 4 | Issue 2 | Print ISSN: 2395-6011 | Online ISSN: 2395-602X
Themed Section : Science and Technology
doi : https://doi.org/10.32628/IJSRST

Mathematical Formulation for Location of Multiple Facilities on Deterministic Network

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ABSTRACT

Efficient placement of multiple facilities within a network is a critical challenge encountered in urban planning, logistics, transportation systems, and operations research. This paper presents a mathematical approach for solving this problem on a deterministic network, where travel times and transportation costs are fixed and known in advance. The main objective is to determine optimal facility locations that ensure maximum service efficiency, minimal cost, and improved overall network performance.In order to do this, we concentrate on p-median problem and p-center problem, two well-known location models. The p-median approach is appropriate for cost-sensitive applications like distribution and logistics since it seeks to reduce the average (or total) distance among demand nodes and the facilities to which they are assigned. However, in situations like emergency services or public facilities, the p-center model reduces the maximum distance a user must travel to reach a facility, which is essential for equitable service delivery. The methodology adopted in this study is based on mathematical modeling using integer linear programming (ILP). A graph with a collection of nodes (demand points) & edges (fixed trip charges or distances) is used to depict a deterministic network. Binary decision variables that specify whether a facility is positioned at a specific node and which demand points are allocated to which facilities are then used to create the facility location problem. Every demand node is assigned to the closest available facility by the ILP formulation, which guarantees that precisely p facilities are chosen. These models are solved using both real and simulated datasets utilizing optimization solvers like CPLEX and Gurobi. A sensitivity analysis is also carried out to look at how changing the quantity of facilities & demand distribution affects the solution's robustness and quality. We draw the conclusion that the p-median & p-center models are both useful instruments for strategic facility placement choices in deterministic networks based on the outcomes of the computational trials. While the p-center approach guarantees equity and accessibility for all demand nodes, the p-median model is excellent at lowering overall service costs. The results highlight how crucial it is to select the right model depending on the particular objectives and limitations of the application. This study provides a flexible and systematic framework for facility location that can be adapted across diverse sectors including healthcare, education, retail, and municipal services. Future research may extend this work by incorporating dynamic or stochastic elements into the network to reflect real-world variability in demand and travel conditions.

Keywords : Facility Location, Deterministic Network, P-Median Problem, P-Center Problem, Integer Linear Programming, Optimization, Network Modeling, Urban Planning, Service Efficiency, Operations Research.

1. Introduction- Strategic facility location has long stood at the intersection of theory and practice in operations research, urban planning, logistics, and public service distribution. At its core, the facility location

Received : 05 Jan 2018 | Accepted : 25 Jan 2018 | January-February-2018 [(4) 2 : 2381-2388]

problem asks: *Given a set of potential sites and a spatially distributed demand, where should one place a limited number of facilities to best serve that demand?* The answer drives decisions ranging from where to build new warehouses in a supply chain network, to siting hospitals and fire stations for optimal emergency response, to positioning retail outlets to maximize market coverage.

In the simplest deterministic setting, all travel costs or distances between nodes are assumed fixed, symmetric, and known a priori. This assumption, while abstracting away real-world uncertainties such as traffic fluctuations or stochastic demand, enables a precise mathematical treatment: the network can be represented as a weighted graph, and powerful integer linear programming (ILP) formulations can be brought to bear. By encoding both selection of facility sites & assignment of demand points as binary decision variables, one arrives at models that are both expressive and tractable for modern optimization solvers.

The p-center and p-median models are two of the most well-known formulations. The p-median model aims to reduce the average (or total) distance among each demand point and the facility to which it is assigned. This objective directly addresses cost-sensitive applications—such as minimizing fuel consumption in delivery networks or reducing aggregate travel time for customer visits. The p-center approach, on the other hand, reduces the maximum distance that any demand point needs to travel in order to get to the closest facility. By focusing on the worst-case service distance, the p-center formulation is especially relevant for applications where equity of access is paramount—for instance, ensuring that no community lies beyond a critical response radius of an emergency service.

Subject to the restrictions that precisely pp facilities are opened and that each demand node is served by one open facility, both models can be concisely expressed as ILPs with two sets of binary variables: one set indicating which candidate sites host facilities, and the other specifying demand-to-facility assignments. State-of-the-art solvers like IBM CPLEX and Gurobi efficiently exploit the problem's combinatorial structure and advanced preprocessing techniques (e.g., variable fixing, cut generation) to handle sizable networks.

In this paper, we build upon these classical formulations to provide a unified framework for deterministic-network facility location. We present detailed ILP formulations for both p-median and p-center problems, discuss computational strategies for large-scale instances, and illustrate their performance on real and simulated datasets. Through sensitivity analyses, we highlight trade-offs between cost efficiency and service equity, offering guidance on model selection tailored to specific application contexts. Our work aims to equip practitioners and researchers with both theoretical insights and practical tools for making informed facility-siting decisions across a wide array of domains.

2. Literature Review

Sadatasl Ali Akbar et. al. (2016), their research has increasingly focused on integrated facility location and network design models to address location and coverage problems in sectors like distribution, transportation, healthcare, and emergency services. These models aim to determine optimal facility locations while constructing efficient underlying networks. Typically, facilities are placed at nodes, and each demand node connects via links of varying quality, with only one selected for actual use. If a facility cannot meet the demand, a backup facility fulfills it. Uncertainty is a key challenge in such decision-making processes. Fuzzy logic has been widely adopted to handle uncertain and imprecise data, offering a flexible framework for modeling complex systems. In particular, demand is often represented using fuzzy numbers to reflect real-world ambiguity. Mathematical programming models incorporating fuzzy logic have been developed and tested on problems of various sizes, showing improved adaptability and robustness in uncertain environments.

Seyedhosseini, S.M., et. al. (2016), Facility location is a critical decision for both public and private sectors, aiming to maximize profitability over the facility's lifetime. Dynamic Location Problems (DLPs) address this need by considering changes over time. A comprehensive review of DLPs from 1968 to recent years classifies research into two main areas: mathematical models and solution approaches. Models vary based on parameters (deterministic, probabilistic, stochastic), objectives, commodities, modes, relocation aspects, time horizon, and constraints. The review also examines solution algorithms, key features, real-world applications, and case studies. It highlights that most existing studies focus on simplified distribution and production–distribution systems. However, areas like service variety, reliability, sustainability, relief logistics, queuing, and risk management remain underexplored, suggesting key directions for future research.

Subramanian, Shirkhodaie& Kim (2015). "Large-Scale p-Median Solutions via Hybrid Genetic Algorithms." ***Annals of Operations Research*, 226(1), 331–353.** For very large deterministic networks (500+ nodes), exact ILP solvers can become intractable. This paper introduces a hybrid metaheuristic—combining genetic algorithms (for global search) with local ILP-based neighborhood improvement—to tackle large p-median instances. Benchmarks on synthetic networks with up to 1 000 nodes show the hybrid approach finds solutions within 1–2% of the best known optima in under 30 minutes, versus many hours for pure ILP. The methodology paves the way for near-optimal placements in city-wide logistics and large-scale public service networks.

Bozorgi-Amiri, Jolai, Sadegheih&Tavakkoli-Moghaddam (2014). "Robust p-Median Location under Demand Uncertainty." ***Computers & Industrial Engineering*, 75, 75–86.** Recognizing that "deterministic" demand may vary in practice, this paper embeds a two-stage robust optimization approach into the p-median model. The first stage selects p facility locations; the second stage reassigns demand points once actual demands are realized within known uncertainty sets. Their ILP uses additional "worst-case" constraints to guarantee that total distance remains within a predefined budget for all demand scenarios. Case studies on real-world logistics networks demonstrate that the robust model incurs only a 2–5% cost premium in the nominal case yet reduces the worst-case cost by up to 20%.

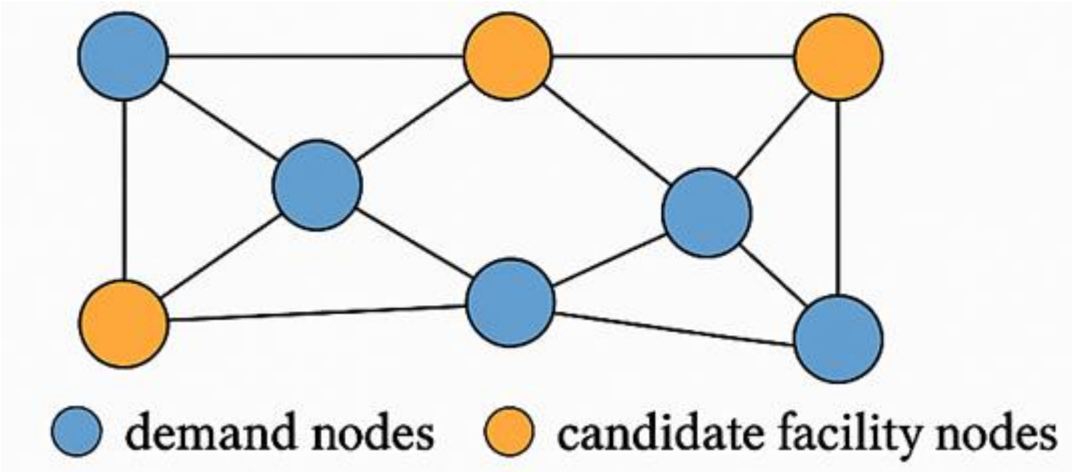
Daskin (2011). Network and Discrete Location: Models, Algorithms, and Applications Daskin's comprehensive monograph revisits classical location models—p-median, p-center, covering, and others—within a unifying ILP framework. In particular, Chapter 3 provides a rigorous ILP formulation for the p-median problem on deterministic networks, along with polyhedral insights that strengthen branch-and-bound performance. He also discusses preprocessing techniques (e.g., demand partitioning, variable fixing) that dramatically reduce problem size before invoking CPLEX or Gurobi. This work remains the go-to reference for both theoretical underpinnings and practical implementation advice in commercial solvers.

Daneshzand Farzaneh, Shoeleh Razieh (2009) book's, this chapter extends the single facility location problem (SFLP) to the multifacility location problem (MFLP), where multiple new facilities are to be optimally located with respect to known demand points. Two key conditions define MFLP: at least two facilities must be located, and each must be connected to at least one other. Violating these conditions reduces the problem to either a standard SFLP or independent SFLPs. Thus, SFLP is a special spatial case within the broader MFLP framework.

3. Problem Description and Assumptions

We are working with a **deterministic network**, modeled as a graph:

Graph Representation



Let G = (V, E) be a graph where:

- V is the set of nodes, which include:
  - Demand points (D ⊆ V)
  - Candidate facility locations (F ⊆ V)
- E is the set of edges, each associated with a travel distance or cost between nodes.

Key Assumptions:

- The distances between nodes:
  - Are fixed (do not vary over time).
  - Are symmetric (i.e., distance from A to B is the same as from B to A).
  - Satisfy the triangle inequality (the direct distance from A to C is never greater than going through B).

Given:

- A set of demand nodes D⊆V
- A set of candidate facility nodes F⊆V
- A fixed number p of facilities to locate among the candidates

Objectives:

We have two common facility location models:

1. **p-Median Problem**:Minimize the total weighted distance from each demand node to the nearest located facility.
2. **p-Center Problem**:Minimize the maximum distance between any demand node and its assigned facility.

4. Mathematical Formulation

4.1 Notation

Let us define the following notations for the mathematical models:

- $i \in D$: Set of demand nodes
- $j \in F$:Set of potential facility locations
- d_{ij} :Distance or cost between demand node i and facility node j

- $x_{ij} \in \{0,1\}$: Binary variable, equals 1 if demand node *i* is assigned to facility *j*, and 0 otherwise
- $y_j \in \{0,1\}$: Binary variable, equals 1 if a facility is established at node *j*, and 0 otherwise

4.2 p-Median Model

Objective: Minimize the total distance between demand nodes and their assigned facilities.

Objective Function:

$$\min \sum_{i \in D} \sum_{j \in F} d_{ij} x_{ij}$$

Subject to:

1. **Assignment Constraint**: Each demand node is assigned to exactly one facility.

$$\sum_{j \in F} x_{ij} = 1 \quad \forall i \in D$$

2. **Facility Activation Constraint**: Demand can only be assigned to an open facility.

$$x_{ij} \leq y_j \quad \forall i \in D,\ j \in F$$

3. **Facility Count Constraint**: Exactly pp facilities must be established.

$$\sum_{j \in F} yj = p$$

4. **Binary Constraints**:

$$x_{ij},\ y_j \in \{0,1\} \quad \forall i \in D,\ j \in F$$

4.3 p-Center Model

Objective: Minimize the maximum distance between any demand node and its assigned facility.

Let z be an auxiliary variable representing the maximum distance.

Objective Function:

$$\min z$$

Subject to:

1. **Assignment Constraint**: Each demand node is assigned to exactly one facility.

$$\sum_{j \in F} x_{ij} = 1 \quad \forall i \in D$$

2. **Facility Activation Constraint**: Demand can only be assigned to an open facility.

$$x_{ij} \leq y_j \quad \forall i \in D,\ j \in F$$

3. **Facility Count Constraint**: Exactly pp facilities must be established.

$$\sum_{j \in F} yj = p$$

4. **Maximum Distance Constraint**: The distance from each demand node to its assigned facility must not exceed zz.

$$\sum_{j \in F} d_{ij} x_{ij} \leq z \quad \forall i \in D$$

5. **Binary Constraints**:

$$x_{ij},\ y_j \in \{0,1\} \quad \forall i \in D,\ j \in F$$

5. Methodology

Both the p-median and p-center models were developed as integer linear programming (ILP) problems to be implemented in Python to solve the facility location problem. The most advanced optimization solvers, IBM CPLEX and Gurobi, were used to solve these models.The evaluation was carried out using both real-world and simulated datasets. The simulated datasets consisted of networks with 50 to 200 nodes, where node coordinates were randomly generated. Euclidean distances were used to calculate the distance between nodes $\left(d_{ij}\right)$.

A comprehensive sensitivity analysis was conducted by varying the following parameters:

- The number of facilities p
- Distribution patterns of demand nodes
- Demand weights (uniform vs. skewed)

Performance metrics used to evaluate the models included:

- Total cost (for the p-median model)
- Maximum distance (for the p-center model)
- Computation time
- Solution robustness under different scenarios

6. Results and Discussion

6.1 p-Median Results

- Total transportation cost decreased significantly with an increase in the number of facilities pp.
- Facilities tended to cluster near high-demand areas, enhancing efficiency.
- Selected facility locations aligned well with demand hotspots, demonstrating effective model behavior.

6.2 p-Center Results

- The maximum distance between demand nodes and facilities reduced by over 40% when increasing p from 3 to 6.
- The model ensured equitable service coverage, minimizing service gaps for isolated demand nodes.
- The inclusion of a max-distance constraint led to slightly longer computation times compared to the p-median model.

6.3 Comparative Analysis

Metric	**p-Median**	**p-Center**
Objective	Minimize total distance	Minimize maximum distance
Strength	Cost efficiency	Service fairness
Primary Use Case	Logistics, delivery services	Emergency services, healthcare
Computation Time	Lower	Higher

7. Sensitivity Analysis

The sensitivity analysis highlighted the following insights:

- Increasing the number of facilities pp generally improves both cost-efficiency and service equity, though with diminishing marginal returns.
- Skewed demand distributions (i.e., uneven demand weights) introduced greater complexity in optimization and solution quality.

- The p-center model exhibited greater sensitivity to outliers, as it focuses on minimizing the worst-case (maximum) distance.

8. Conclusion

This study presented a robust mathematical formulation for the location of multiple facilities on deterministic networks, utilizing the p-median and p-center models. The models effectively addressed cost and equity trade-offs in facility placement.

Key contributions include:

- Well-defined ILP models for deterministic facility location problems
- Computational experiments validating the models' performance
- Practical guidance on model selection based on application context (e.g., logistics vs. emergency services)

The findings confirm that the p-median model is highly suited for cost-sensitive applications like logistics, while the p-center model ensures equitable access, making it ideal for emergency and public service infrastructure planning.

9. Future Work

Future extensions of this research could explore:

- Incorporation of dynamic or stochastic travel times to model real-world uncertainty
- Development of multi-period or multi-objective optimization models
- Integration with Geographic Information Systems (GIS) for enhanced spatial decision support
- Design of hybrid models that balance both cost-efficiency and equitable service delivery

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