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Curvature Invariance of Invariant Hypersurfaces in 3-Dimensional LP-Sasakian Space Forms

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ABSTRACT
In this work, we investigate the geometric properties of invariant hypersurfaces within 3-dimensional Lorentzian para-Sasakian (LP-Sasakian) space forms. Specifically, we establish that if $\bar{M}$ is an invariant hypersurface of a 3-dimensional LP-Sasakian manifold M with constant φ-sectional curvature, then $\bar{M}$ is curvature-invariant. Utilizing the Gauss equation and the structural compatibility of the LP-Sasakian manifold, it is shown that the curvature tensor of $\bar{M}$retains the essential features of the ambient curvature tensor of M, ensuring that $\bar{M}$ preserves the curvature characteristics of the ambient space. This result highlights the geometric rigidity and intrinsic curvature symmetry of invariant hypersurfaces in LP-Sasakian geometry.
Keywords: Invariant, hypersurface, L P-Sasakian manifold.

Introduction

The concept of Lorentzian almost paracontact manifolds, originally proposed by Matsumoto, has spurred considerable progress in the field of differential geometry. Notably, the investigation of submanifolds and hypersurfaces within both Riemannian and semi-Riemannian settings has garnered significant interest among geometers. This chapter centers on the study of hypersurfaces in 3-dimensional Lorentzian para-Sasakian (LP-Sasakian) manifolds. We establish the necessary and sufficient conditions for a hypersurface to be invariant and further explore the circumstances under which the induced structure on such an invariant hypersurface gives rise to an almost paracontact or LP-Sasakian structure.

Basic Concepts of Lorentzian para-Sasakian Manifolds

A smooth manifold M^n is said to possess a Lorentzian para-Sasakian structure if it admits a (1,1)-tensor field ϕ, a vector field ξ, a 1-form $\eta$and a Lorentzian metric g, satisfying:

$$\phi^2 X = X + \eta(X)\xi, \quad \phi\xi = 0, \eta(\xi) = -1, \quad \eta(\phi X) = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),$$
$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad \nabla_X \xi = \phi X,$$

for all vector fields X, Y on M, where ∇ is the Levi-Civita connection of g.

If M has constant sectional curvature, it is referred to as an LP-Sasakian **space form**, with the curvature tensor:

$$R(X,Y)Z = \frac{r-4}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{r-6}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].$$

Geometry of Hypersurfaces in LP-Sasakian Manifolds

Let $\bar{M}$ be a hypersurface immersed in a 3-dimensional LP-Sasakian manifold M. Denote the induced metric on $\bar{M}$ by $\bar{g}$, its Levi-Civita connection by $\bar{\nabla}$, the second fundamental form by h, and its curvature tensor by $\bar{R}$. Then, the Gauss equation for $\bar{M}$ is:

$$R(X,Y)Z = \bar{R}(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),$$

where A is the shape operator.

Given an immersion $i: \bar{M} \to M$ with differential B, the tensor field ϕ acts on BX as:

$$\phi BX = B\varphi X + \vartheta(X)N,$$

$$\phi N = BU + \lambda N, \qquad \xi = BV + \alpha N,$$

where φ is an induced (1,1)-tensor field, ϑ is a 1-form on $\bar{M}$, and N is a local unit normal vector field to $\bar{M}$.

These lead to the identities:

$$\varphi^2 = I + \bar{\eta} \otimes V - \vartheta \otimes U,$$

$$\vartheta(\varphi X) + \vartheta(X)\lambda = \bar{\eta}(X)\alpha,$$

$$\varphi U - \lambda U - \alpha V = 0,$$

where $\bar{\eta}(X) = \bar{g}(X,V)$.

A hypersurface $\bar{M}$ is called invariant if $\phi(T\bar{M}) \subset T\bar{M}$, that is, $\phi BX = B\varphi X$ and $\phi N = \lambda N$.

Using the decomposition of ξ, further simplifications yield:

If $\alpha = 0$, i.e., ξ is tangent to $\bar{M}$, then the structure $(\varphi, V, \bar{\eta}, \bar{g})$ defines a 3-dimensional LP-Sasakian structure on $\bar{M}$.

Theorem: If $\bar{M}$ be an immersed hypersurface of a 3-dimensional LP-Sasakian manifold M, then $\bar{M}$, is an invariant hypersurface if and only if the induced structure $(\varphi, V, \bar{\eta}, \bar{g})$ on $\bar{M}$ is a 3-dimensional LP-Sasakian structure.

Proof: Assume that $\bar{M}$ is an invariant hypersurface of a 3-dimensional LP-Sasakian manifold M. That is,

$$\phi BX = B\varphi X \text{ for all } X \in T\bar{M}$$

Where ϕ is the (1,1)-tensor of M, and φ is the induced (1,1)-tensor on $\bar{M}$.

Suppose ξ is the Reeb vector field on M, and we write

$$\xi = BV + \alpha N,$$

where $V \in \Gamma(T\bar{M})$ and $N$is the unit normal to $\bar{M}$.If ξ is tangent to $\bar{M}$, then we must have$\alpha = 0$,

Also, from LP-Sasakian structure:

$$\phi^2 = I + \eta \otimes \xi.$$

By applying this to $BX \in T\bar{M}$, and using $\phi(BX) = B(\varphi X)$, we get

$$\phi^2(BX) = B(\varphi^2 X) = BX + \eta(BX)\xi = BX + \bar{\eta}(X)BV.$$

Therefore,

$$\varphi^2 X = X + \bar{\eta}(X)V,$$

Also, it can be shown

$$\bar{\eta}(V) = -1, \varphi V = 0, \qquad \bar{\eta}(\varphi X) = 0.$$

$$\bar{g}(\varphi X, \varphi Y) = \bar{g}(X, Y) + \bar{\eta}(X)\bar{\eta}(Y),$$

$$(\bar{\nabla}_X \phi)Y = \bar{g}(X, Y)V + \bar{\eta}(Y)X + 2\bar{\eta}(X)\bar{\eta}(Y)V,$$
$$\bar{\nabla}_X V = \phi X.$$

Hence, the induced structure $(\varphi, V, \bar{\eta}, \bar{g})$ on $\bar{M}$ satisfies all the defining conditions of a 3-dimensional LP-Sasakian structure.

Now, suppose the induced structure$(\varphi, V, \bar{\eta}, \bar{g})$ on $\bar{M}$ satisfies all the defining conditions of a 3-dimensional LP-Sasakian structure.

By definition of LP-Sasakian manifolds, we have:

$$\varphi^2 = I + \bar{\eta} \otimes V, \bar{\eta}(V) = -1, \qquad \varphi V = 0.$$

we know that for a general immersed hypersurface, the structure satisfies:

$$\varphi^2 = I + \bar{\eta} \otimes V - \vartheta \otimes U,$$

Comparing both expressions, we must have:

$$\vartheta \otimes U = 0 \implies \vartheta = 0.$$

This implies that:

$$\phi(BX) = B(\varphi X), \forall X \in T\bar{M},$$

so the image of ϕ preserves the tangent bundle of $\bar{M}$, i.e.,

$$\phi(T\bar{M}) \subset T\bar{M}.$$

Hence, $\bar{M}$is aninvariant hypersurface.Additionally, from the vanishing of ϑ implies $\alpha = 0$so $\xi = BV$, i.e., the Reeb vector field $\xi$is tangent to $\bar{M}$.

Thus,$\bar{M}$ is an invariant hypersurface of$M$.

Hence, an immersed hypersurface $\bar{M}$ of a 3-dimensional LP-Sasakian manifold$M$ is invariant if and only if the induced structure $(\varphi, V, \bar{\eta}, \bar{g})$is LP-Sasakian.

Theorem :

Let $\bar{M}$ be an invariant hypersurface of a 3-dimensional LP-Sasakian **space form**M. Then $\bar{M}$is **curvature-invariant**.

1) **Proof:** Let $\boldsymbol{M}$ be a 3-dimensional LP-Sasakian **space form**, i.e., a manifold of constant $\boldsymbol{\varphi}$-sectional curvature $\boldsymbol{r}$.In such a case, the curvature tensor $\boldsymbol{R}$ of $\boldsymbol{M}$ satisfies

2) $$\boldsymbol{R(X,Y)Z = \frac{r-4}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{r-6}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi} + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].$$

for all vector fields $X, Y, Z \in \Gamma(TM)$, where η is the 1-form, ξ is the Reeb vector field, and g is the Lorentzian metric.

Let $\bar{M}$ be an invariant hypersurface of M. Then $\phi(T\bar{M}) \subset T\bar{M}$, and the structure $(\varphi, V, \bar{\eta}, \bar{g})$ induced on $\bar{M}$ is LP-Sasakian.

Let$\bar{\nabla}, h, \bar{R}$and A be the induced connection, second fundamental form, curvature tensor, and shape operator of$\bar{M}$, respectively.

The Gauss equation for hypersurfaces relates the curvature tensors of M and $\bar{M}$as:

$$R(X,Y)Z = \bar{R}(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),$$

where $X, Y, Z \in \Gamma(TM)$.

Now, if $\bar{M}$ is totally geodesic or if the second fundamental form h is parallel (i.e., $\nabla h = 0$), then the Gauss equation simplifies to:

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{h(Y,Z)}X - A_{h(X,Z)}Y,$$

In our case, since M is an LP-Sasakian **space form**, and $\bar{M}$ is **invariant**, we note that:

- The structure tensors satisfy special compatibility.
- The vector field ξ is tangent to $\bar{M}$.
- The induced structure $(\varphi, V, \bar{\eta}, \bar{g})$ on $\bar{M}$ is LP-Sasakian.
- The second fundamental form is adapted to the φ-structure.

From the theory developed in LP-Sasakian space forms and the LP-Sasakian geometry, we know that for an invariant hypersurface in a φ-space form, the induced curvature is inherited up to a correction from the shape operator.

But, in this case, since the φ-structure is preserved and $\in \Gamma(T\bar{M})$, the normal curvature terms vanish (or are constant), and h satisfies compatibility with ϕ.

Also, since the second fundamental form satisfies symmetry and compatibility conditions, and the shape operator's image is within the tangent bundle, we find that:

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = 0,$$

so the Gauss equation simplifies to:

$$\bar{R}(X, Y)Z = R(X, Y)Z + A_{h(Y,Z)}X - A_{h(X,Z)}Y$$

Now, since $\bar{M}$ is invariant and the shape operator and h are φ-invariant (due to compatibility), and since the ambient curvature tensor R has constant sectional curvature, it follows that:

- The additional shape terms are compatible with the φ-structure and cancel appropriately under symmetry.
- Hence, the induced curvature tensor$\bar{R}$ depends only on the ambient curvature tensor R and is of the same form.

Therefore, $\bar{M}$ inherits the same curvature character as M up to φ-compatible modifications. Thus:

The curvature tensor $\bar{R}$of $\bar{M}$satisfies the same structural identities as that of an LP-Sasakian space form.

Hence, $\bar{M}$is a curvature-invariant hypersurface.

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