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Fixed points and Stability of Damped and Unforced Duffing Equation, Solution and Graphs for Displacement and Velocity of that Equation using Matlab

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ABSTRACT

The object of the present paper is to find the fixed points and their stability by **matrix method** of the duffing equation (damped ($\delta \neq 0$) and unforced / undriven
($\gamma = 0$)) $\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\cos(\omega t)$,
$x = x(t)$, $t = time\ period$ (1)
using special case $\nu = 1 = \rho$ and taking the minus sign in (1), it becomes $\ddot{x} + \delta\dot{x} + (x^3 - x) = 0$
The duffing equation can be expressed in $y_2(t)$ and $\dot{y}_2(t)$ shown by MATLAB program and their graphs.

Keywords : Duffing Equation, Jacobi Elliptic Functions, Period, Boundedness

I. INTRODUCTION

The concept of Duffing equation or Duffing Oscillator was named after Georg Duffing (1861 - 1944), is a non -linear second order differential equation used to model certain damped ($\delta \neq 0$) and driven ($\gamma \neq 0$) oscillators.

The equation is given by

$$\ddot{x} + \delta\dot{x} + (\rho x \pm \nu x^3) = \gamma\cos(\omega t) \quad (1)$$

where $x(t)$ = displacement at time t

$\dot{x} = \frac{d}{dt}(x(t))$ = velocity

and $\ddot{x} = \frac{d^2}{dt^2}(x(t))$ = acceleration.

The numbers $\delta, \rho, \nu, \gamma, \omega$ are given constants.

Parameters

The parameters of equation (1) are

- δ controls the amount of damping
- ρ controls the linear stiffness
- ν controls the amount of non-linearity
- γ is the amplitude of the periodic driving force
- ω is the angular frequency of the periodic driving force

Note

- If $\nu = 0$, then (1) describes the damped and
- driven simple harmonic oscillator.
- If $\gamma = 0$, then (1) describes the undamped
- oscillator.
- The restoring force provided by the non-linear spring is $\rho x + \nu x^3$ (2)

Case – I

If $\rho > 0$ and $\nu > 0$, then (2) is called a hardening spring

Case – II

If $\rho > 0$ and $\nu < 0$, then (2) is called a softening spring

The number of parameters in (1) can be reduced to two through scaling by the excursion x and the time t. Now, let $\tau = t\sqrt{\rho}$ and $y = \frac{x\rho}{\gamma}$, $\rho > 0$ and substituting it in (1), we get $\ddot{y} + 2\eta\dot{y} + y + \epsilon y^3 = \cos(\omega\tau)$

where $\eta = \frac{\delta}{2\sqrt{\rho}}$, $\epsilon = \frac{\nu\gamma^2}{\rho^2}$, $\sigma = \frac{\omega}{\sqrt{\rho}}$

and the dots denote differentiation of $y(\tau)$ with respect to τ.

This shows that the solutions to (1) can be described in terms of the three parameters $\eta,\ \epsilon,\ \sigma$ with two initial conditions $y(t_0)$ and $\dot{y}(t_0)$.

Equilibrium Points

The equilibrium points stable and unstable are at (2)

Case - I

If $\rho > 0$, then the equilibrium point is at $x = 0$.

Case - II

If $\rho < 0$ and > 0 , then the equilibrium points are at

$x = \pm\sqrt{-\frac{\rho}{\nu}}$

Methods of Solution

Many approximate solutions for the duffing equation are

- By Fourier series method.
- By Frobenius method which yields a complex solution.
- By numeric methods such as Euler's method and Runge-Kutta.
- By Homotopy analysis method which yields approximate solutions of the duffing equation.
- In the special case of the undamped (δ = 0) and undriven ($\gamma = 0$) duffing equation an exact solution can be obtained using Jacobi's elliptic functions.

II. DAMPED AND UNFORCED DUFFING EQUATION BY MATRIX METHOD

The duffing equation is

$\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\cos(\omega t),$

$x = x(t),\ t = time\ perio$ (1)

In this chapter, my aim is to find the fixed points and their stability by **matrix method** of the duffing equation (damped ($\delta \neq 0$) and unforced / undriven ($\gamma = 0$)) using special case $\nu = 1 = \rho$ and taking the minus sign in (1) becomes

$$\ddot{x} + \delta\dot{x} + (x^3 - x) = 0 \qquad (2)$$

Now, (2) transforms to a system of first order differential equations. (See [4], [10])

Setting

$$\dot{x} = y \qquad (3)$$

$$\dot{y} = x - x^3 - \delta y \qquad (4)$$

Now, for the fixed point, the system of differential equations are

$\dot{x} = y = 0$

and $\dot{y} = x - x^3 - \delta y$

$\Rightarrow\ 0 = x(1 - x^2) - 0$

$\Rightarrow\ 0 = x(1 - x^2)$

$\Rightarrow\ x(1 - x^2) = 0$

$\Rightarrow\ x = 0 \quad or \quad (1 - x^2) = 0$

$\Rightarrow\ x = 0 \quad or \quad x^2 = 1$

$\Rightarrow\ x = 0 \quad or \quad x = \pm 1$

$\therefore$ The fixed points are $(-1,\ 0),\ (1,\ 0)\ and\ (0,\ 0)$.

Analysis of the stability of the fixed points by matrix method to be determined.

Differentiating (3) and (4) gives

$\ddot{x} = \dot{y}$

$\ \ = x - x^3 - \delta y$

$\ddot{y} = (1 - 3x^2)\dot{x} - \delta\dot{y}$

Which can be written as the matrix equation

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ (1-3x^2) & -\delta\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$$

Case - I (for (0, 0))

The characteristic equation is

$$\det(A - \lambda I) = 0,$$

λ = characteristic values which to be determined

$$\det\left(\begin{bmatrix}0 & 1\\ (1-0) & -\delta\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix}0-\lambda & 1-0\\ 1-0 & -\delta-\lambda\end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix}-\lambda & 1\\ 1 & -\delta-\lambda\end{bmatrix}\right) = 0$$

$$\Rightarrow\ -\lambda(-\delta - \lambda) - 1 = 0$$

$\Rightarrow \lambda^2 + \lambda\delta - 1 = 0$

$$\Rightarrow \lambda = \frac{\left(-\delta \pm \sqrt{\delta^2 - 4 \times (-1) \times 1}\right)}{2 \times 1}$$

$$\Rightarrow \lambda_{0,\ 0} = \frac{1}{2}\left(-\delta \pm \sqrt{\delta^2 + 4}\right)$$

Since, $\delta^2 \geq 0$

So, $\lambda_{0\,,0}$ is real.

Since, $\sqrt{\delta^2+4} > |\delta|$

λ has one positive root only.

So, this fixed point (0, 0) is unstable.

Case II (for $(\pm 1,\ 0)$)

The characteristic equation is

$$\det\left(\begin{bmatrix} 0 & 1 \\ (1-3) & -\delta \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 0-\lambda & 1-0 \\ -2-0 & -\delta-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -\delta-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow -\lambda(-\delta-\lambda)+2=0$$

$$\Rightarrow \lambda^2 + \lambda\delta + 2 = 0$$

$$\Rightarrow \lambda = \frac{\left(-\delta \pm \sqrt{\delta^2 - 4\times 2\times 1}\right)}{2\times 1}$$

$$\Rightarrow \lambda_{\pm 1,\ 0} = \frac{1}{2}\left(-\delta \pm \sqrt{\delta^2-8}\right)$$

For $\delta > 0$

So, $\lambda_{\pm 1,\ 0}$ is imaginary.

λ has complex roots.

So, the point $(\pm 1, 0)$ is asymptotically stable.

Case III (for Undamped $(\delta = 0)$)

$$\lambda_{\pm 1,\ 0} = \frac{1}{2}\left(-\delta \pm \sqrt{\delta^2-8}\right)$$

$$\Rightarrow \lambda_{\pm 1,\ 0} = \frac{1}{2}\left(0 \pm \sqrt{0-8}\right)$$

$$\Rightarrow \lambda_{\pm 1,\ 0} = \frac{1}{2}\left(0 \pm \sqrt{-8}\right)$$

$$\Rightarrow \lambda_{\pm 1,\ 0} = \frac{1}{2}\left(0 \pm i2\sqrt{2}\right)$$

$$\Rightarrow \lambda_{\pm 1,\ 0} = \pm i\sqrt{2}$$

Since, $\delta = 0$

So, the fixed point $\lambda_{\pm 1,\ 0}$ has an imaginary root and is linearly stable. See [10]

Case IV for $(\delta \in (-2\sqrt{2}, 0))$

The fixed point $\lambda_{\pm 1,\ 0}$ has a real root and is unstable.

Case V for $(\delta = -2\sqrt{2})$

$\lambda_{\pm 1,\ 0} = \sqrt{2}$

The fixed point $\lambda_{\pm 1,\ 0}$ has a positive real root and is unstable.

Case VI for $(\delta < -2\sqrt{2})$

The fixed points $\lambda_{\pm 1,\ 0}$ are positive real roots and are unstable.

III. SOLUTIONS AND GRAPHS OF ISPLACEMENT AND VELOCITY FOR DUFFING EQUATION USING MATLAB

Consider the duffing equation

$$\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\cos(\omega t) \qquad (1)$$

where $x = x(t),\ \ t = time\ period$

Substituting $y_1(t) = x(t)$ *and* $y_2(t) = \dot{x}(t)$ and plus sign in (1), we get

$$\dot{y}_1(t) = \dot{x}(t) = y_2(t)$$

$$\dot{y}_2(t) = \ddot{x}(t)$$

$$= -\delta y_2(t) - \rho y_1(t) - \nu y_1^3(t) + \gamma\cos(\omega t)$$

Consider the parameters $\rho = 1,\ \nu = -1,\ \gamma = 3,\ \delta = 2,\ \omega = 1.$

In this chapter, the duffing equation can be expressed in $y_2(t)$ and $\dot{y}_2(t)$ shown by MATLAB program and their graphs. (see [4], [1], [10])

Program :

M-file

```
function dydt = duffing_2(t, y)
%Local parameters
rho = 1;
nu = -1;
gamma = 3;
delta = 2;
omega = 1;
%State vector is [y(t); ydot(t)];
dydt(1) = y(2);
dydt(2) = -delta*y(2)-rho*(y(1))-
    nu*(y(1)^3)+gamma*cos(omega*t);
dydt = dydt';
return
```

Command Window :

```
>> figure
>> [t, y] = ode45(@duffing_2, [0 , 5],
        [0 ; 0]);
>> p = plot(t, y, '*')
>> title('DUFFING EQUATION : \rho = 1,
```

```
    \nu = -1, \gamma = 3,
    \delta = 2, \omega = 1')
>> xlabel('Time (t) in sec')
>> ylabel('Solution y2(t) and
     Derivative of y2(t)')
>> legend('y2(t)', ‘Derivative of y2(t)')
```

IV. Result

t =

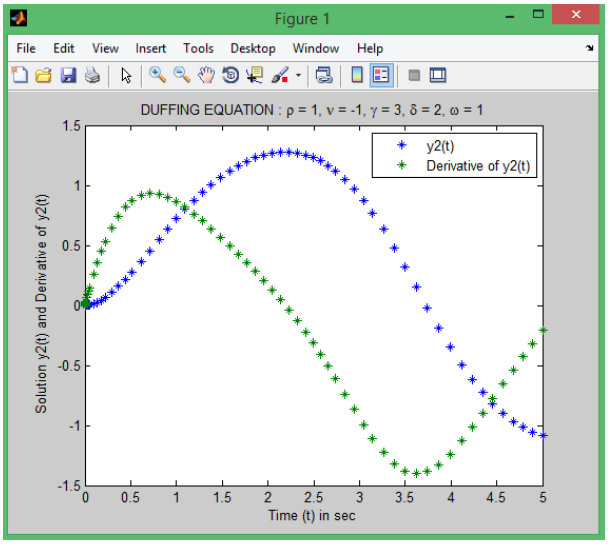
0	0.2953	2.5766
0.0000	0.3686	2.6587
0.0000	0.4419	2.7409
0.0001	0.5152	2.8418
0.0001	0.6129	2.9428
0.0002	0.7106	3.0438
0.0002	0.8083	3.1448
0.0003	0.9059	3.2631
0.0004	1.0001	3.3814
0.0008	1.0943	3.4997
0.0012	1.1885	3.6179
0.0017	1.2827	3.7429
0.0021	1.3813	3.8679
0.0042	1.4799	3.9929
0.0063	1.5785	4.1179
0.0084	1.6771	4.2313
0.0104	1.7690	4.3447
0.0209	1.8609	4.4580
0.0314	1.9528	4.5714
0.0418	2.0447	4.6785
0.0523	2.1367	4.7857
0.0947	2.2286	4.8928
0.1372	2.3205	5.0000
0.1796	2.4124	
0.2220	2.4945	

y =

0	0	0.9424	0.7023
0.0000	0.0001	1.0084	0.6358
0.0000	0.0001	1.0677	0.5666
0.0000	0.0002	1.1201	0.4956
0.0000	0.0002	1.1653	0.4228
0.0000	0.0005	1.2010	0.3533
0.0000	0.0007	1.2303	0.2818
0.0000	0.0010	1.2528	0.2078
0.0000	0.0012	1.2683	0.1308
0.0000	0.0025	1.2767	0.0498
0.0000	0.0037	1.2774	-0.0357
0.0000	0.0050	1.2700	-0.1261
0.0000	0.0062	1.2540	-0.2220
0.0000	0.0125	1.2321	-0.3126
0.0001	0.0187	1.2026	-0.4076
0.0001	0.0249	1.1650	-0.5066
0.0002	0.0310	1.1192	-0.6088
0.0006	0.0614	1.0513	-0.7376
0.0014	0.0912	0.9704	-0.8665
0.0026	0.1204	0.8765	-0.9915
0.0040	0.1489	0.7703	-1.1081
0.0126	0.2581	0.6320	-1.2273
0.0257	0.3575	0.4811	-1.3194
0.0428	0.4476	0.3212	-1.3780
0.0636	0.5286	0.1565	-1.3995
0.1068	0.6484	-0.0176	-1.3817
0.1580	0.7444	-0.1870	-1.3256
0.2154	0.8184	-0.3475	-1.2371
0.2775	0.8724	-0.4957	-1.1260
0.3651	0.9169	-0.6170	-1.0143
0.4556	0.9337	-0.7252	-0.8962
0.5467	0.9272	-0.8200	-0.7755
0.6363	0.9021	-0.9012	-0.6548
0.7196	0.8649	-0.9654	-0.5419
0.7988	0.8175	-1.0175	-0.4298
0.8733	0.7624	-1.0575	-0.3179
		-1.0854	-0.2048

p =
174.0133
175.0128

Graph :



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