International Journal of Scientific Research in Science and Technology
Print ISSN: 2395-6011 | Online ISSN: 2395-602X (www.ijsrst.com)
doi : https://doi.org/10.32628/IJSRST207524

On Some Special Finsler Spaces

Ankit Maurya*, K.B. Gupta**, Jitendra Singh***

*Research Scholar in Mathematics, Singhania University, Pacheri Bari (Jhunjhunu), Rajasthan, India

**Department of Mathematics, St. Andrew's P.G. College, Gorakhpur, Uttar Pradesh, India

***Department of Physics, Sri. Lal Bahadur Shastri Degree College, Gonda, Uttar Pradesh, India

Article Info
Volume 7, Issue 5
Page Number: 76-87
Publication Issue :
September-October-2020

Article History
Accepted :10 Sep 2020
Published :20 Sep2020

ABSTRACT

The present communication has mainly been divided into four sections of which the first section is introductory, the second section deals with R$^+$-recurrent F_n^* of order one. In this section we have derived results telling as to when a $^+R^i_{jk}$ - ⊕ recurrent F_n^* of order one will be R$^+$ - ⊕ recurrent of order one, $^+R^i_j$ - ⊕ recurrent F_n^* of order one will be a $^+R^i_{jk}$ -⊕ recurrent of order one. In this section we have also derived the Bianchi's identity and few more identities which hold in a R$^+$- recurrent F_n^* of order one. The third section deals with R$^+$- recurrent F_n^* of order two. In this section we have observed that the recurrence tensor field $b_{lm}(x,\dot{x})$ is non-symmetric, few more relations and the Bianchi's identity have been derived in a R$^+$-recurrent F_n^* of order two. In the fourth and the last section we have derived the conditions under which a Landsberg space in a P_λ-Finsler space, a P_λ-Finsler space is semi - P2- like, a P^*- Finsler space is a P_λ- Finsler space, a P_λ – Finsler space is P- symmetric, a P_λ - Finsler space is P2 like.

Keywords: Recurrent Finsler space, Bianchi's identity, non-symmetric, Landsberg space, semi-P2- like, P* - Finsler space, P_λ – Finsler space, P-symmetric.

I. INTRODUCTION

A non-symmetric connection in an n- dimensional space A_n has been introduced by Vranceanu through his communication [12], we have extended this concept to the theory of n- dimensional Finsler space with non-symmetric connection $\Gamma_{jk}^{i}(\neq \Gamma_{kj}^{i})$ based on a non-symmetric fundamental tensor $g_{ij}(x,\dot{x}) \neq (g_{ji}(x,\dot{x}))$. we write

(1.1) $\Gamma_{jk}^{i} = M_{jk}^{i} + \frac{1}{2}N_{jk}^{i}$,

where M_{jk}^{i} and $\frac{1}{2}N_{jk}^{i}$ are respectively the symmetric and skew-symmetric parts of the non-symmetric connection Γ_{jk}^{i} .as given by Cartan [2], let a vertical stroke (|) followed by an index stand for the covariant derivative with respect to x, thus the covariant derivative of an arbitrary contravariant vector field $X^{i}(x,\dot{x})$ is defined as under:

(1.2) $X^{i\,+}|_{\,j} = \partial j X^{i} - (\dot{\partial}_{m}X^{i})\Gamma_{kj}^{m}\dot{x}^{k} + X^{k}\Gamma_{kj}^{i}$

The covariant derivative defined in this manner shall be called $\oplus$- covariant differentiation of $X^{i}(x,\dot{x})$ with respect to $\dot{x}^{j}$. Differentiating (1.2) $\oplus$- covariantly with respect to $\dot{x}^{k}$ and commutating the result thus obtained with respect to the indices j and k, we shall have the following commutation formula

(1.3) $X^{i\,+}|_{\,[jk]} = -(\dot{\partial}_{m}X^{i})R_{pjk}^{m}\dot{x}^{p} + X^{m}R_{mjk}^{i} + X^{i\,+}|_{\,m}N_{kj}^{m}$

where (1.4) ${}^{+}R_{ijk}^{h} \overset{\text{def}}{=} \partial_{k}\Gamma_{ij}^{h} - \partial_{j}\Gamma_{ik}^{h} + \dot{\partial}_{m}\Gamma_{ik}^{h}\Gamma_{sj}^{m}\dot{x}^{s} - \dot{\partial}_{m}\Gamma_{ij}^{h}\Gamma_{sk}^{m}\dot{x}^{s} + \Gamma_{ij}^{p}\Gamma_{pk}^{h} - \Gamma_{ik}^{p}\Gamma_{pj}^{h}$.

From here onwards the Finsler space equipped with non-symmetric connection will be denoted by F_{n}^{*} and therefore the entities defined by (1.4) shall be called "Curvature tensor" of the Finsler space F_{n}^{*}. The following identities, notations and contractions actually exist in an F_{n}^{*}, which we shall extensively use in the later discussion:

(1.5) (a) $x^{i+}|_{\,k} = \dot{x}^{i\,-}|_{\,k} = 0$, (b) $R_{jk}^{i} = R_{hjk}^{i}\dot{x}^{h}$, (c) $R_{j}^{i} = R_{hj}^{i}\dot{x}^{h}$,

(d) $R_{hjk}^{i} = -R_{hkj}^{i}$, (e) $R_{i}^{i} = (n-1)\,R$,

(f) $N_{jk}^{i} = -N_{kj}^{i} = \Gamma_{jk}^{i} - \Gamma_{kj}^{i}$, (g) $\Gamma_{hjk}^{i} = \dot{\partial}_{h}\Gamma_{jk}^{i}$.

2. R^{+}- RECURRENT F_{n}^{*} OF ORDER ONE:

We propose the following definition which shall be found useful in the later discussions.

DEFINITION(2.1):

A Finsler space F_{n}^{*} equipped with non-symmetric connection shall be called R^{+}- recurrent of order one if

(2.1) ${}^{+}R_{ijk}^{h}{}^{+}|_{\,l} = p_{l}\,{}^{+}R_{ijk}^{h}$,

where ${}^{+}R_{ijk}^{h}(x,\dot{x})$ is the curvature tensor in F_{n}^{*} as has been given by (1.4) and the $p_{l}(x,\dot{x})$ is the non-null recurrence vector.

We now transvect (2.1) successively by $\dot{x}^{i}$ and $\dot{x}^{j}$ and there after use the set of relations given in (1.5) and get

(2.2) ${}^{+}R_{jk}^{h}\,{}^{+}|_{\,l} = p_{l}\,{}^{+}R_{jk}^{h}$

and (2.3) ${}^{+}R_{k}^{h}\,{}^{+}|_{\,l} = p_{l}\,{}^{+}R_{k}^{h}$.

the observations given in (2.2) and (2.3) enable us to state that the tensor fields ${}^{+}R_{jk}^{h}$ and ${}^{+}R_{k}^{h}$ are also $\oplus$- recurrent of order one in a R^{+}- recurrent F_{n}^{*} having the same recurrence vector. under such a circumference the Finsler space F_{n}^{*} is respectively termed as ${}^{+}R_{jk}^{h}$ - $\oplus$ recurrent and ${}^{+}R_{k}^{h}$ - $\oplus$ recurrent of one but converse of this statement is not always found to be true. We now differentiating (2.2) partially with respect to $\dot{x}^{h}$ and commute the result thus obtained with respect to the identities j and k and thereafter use the commutation formula as has been given in (1.3) along with the identities as have been given in (1.5), we get

$$(2.4)\ {}^{+}R^{i}_{hjk}{}^{+}|_{l} - p_l {}^{+}R^{i}_{hjk} = {}^{+}R^{i}_{jk}\dot{\partial}_h p_l + p_l \dot{x}^q \dot{\partial}_h {}^{+}R^{i}_{qjk} - \dot{x}^q(\dot{\partial}_h {}^{+}R^{i}_{qjk})^{+}|_{l} - {}^{+}R^{m}_{jk}\Gamma^{i}_{hml} + {}^{+}R^{i}_{mk}\Gamma^{m}_{hjl} + {}^{+}R^{i}_{jm}\Gamma^{m}_{hkl} + \Gamma^{m}_{hql}\dot{x}^q\dot{\partial}_m {}^{+}R^{i}_{jk}\ .$$

The left hand side of (2.4) vanishes in a R^{+}- recurrent $F_n^*$of order one and hence we get

$$(2.5)\quad \dot{x}^q(\dot{\partial}_h {}^{+}R^{i}_{qjk})^{+}|_{l} = {}^{+}R^{i}_{jk}\dot{\partial}_h p_l + p_l \dot{x}^q \dot{\partial}_h {}^{+}R^{i}_{qjk} - {}^{+}R^{m}_{jk}\Gamma^{i}_{hml} + {}^{+}R^{i}_{mk}\Gamma^{m}_{hjl} + {}^{+}R^{i}_{jm}\Gamma^{m}_{hkl} + \Gamma^{m}_{hql}\dot{x}^q\dot{\partial}_m {}^{+}R^{i}_{jk}\ .$$

In accordance with (2.5), we can therefore state:

<u>THEOREM (2.1)</u>:

In order that an ${}^{+}R^{i}_{jk}$ - ⊕ recurrent F_n^* of order one be R^{+}- ⊕- recurrent of order one it is necessary and sufficient that (2.5) holds.

We now allow a transvection of (2.4) by $\dot{x}^h$ and then use the identities as have been in (1.5), we get

$$(2.6)\quad {}^{+}R^{i}_{jk}{}^{+}|_{l} - p_l {}^{+}R^{i}_{jk} = {}^{+}R^{i}_{jk}\dot{x}^h\dot{\partial}_h p_l + p_l \dot{x}^q\dot{x}^h \dot{\partial}_h {}^{+}R^{i}_{qjk} - \dot{x}^q\dot{x}^h(\dot{\partial}_h {}^{+}R^{i}_{qjk})^{+}|_{l} - {}^{+}R^{m}_{jk}(\dot{\partial}_h \Gamma^{i}_{ml})\dot{x}^h + {}^{+}R^{i}_{mk}(\dot{\partial}_h \Gamma^{m}_{jl})\dot{x}^h + {}^{+}R^{i}_{jm}(\dot{\partial}_h \Gamma^{m}_{kl})\dot{x}^h + (\dot{\partial}_m {}^{+}R^{i}_{jk})(\dot{\partial}_h \Gamma^{m}_{ql})\dot{x}^h\dot{x}^q\ .$$

We now take into account the homogeneity property of the connection parameter along with the fact that in a ${}^{+}R^{i}_{jk}$ - ⊕ recurrent F_n^* of order one the left hand side of (2.6) vanishes automatically, we get

$$(2.7)\quad \dot{x}^q\dot{x}^h(\dot{\partial}_h {}^{+}R^{i}_{qjk})^{+}|_{l} = {}^{+}R^{i}_{jk}\dot{x}^h(\dot{\partial}_h p_l) + p_l \dot{x}^q\dot{x}^h \dot{\partial}_h {}^{+}R^{i}_{qjk}\ .$$

Therefore, we can state:

<u>THEOREM (2.1)</u>:

In a ${}^{+}R^{i}_{jk}$ - ⊕ recurrent F_n^* of order one the identity as has been given by (2.7) always holds under the condition that the connection parameter $\Gamma^{i}_{jk}(x, \dot{x})$ be assumed to be homogeneous of degree zero in its directional arguments.

We now consider the case when the Finsler space F_n^* under consideration in ${}^{+}R^{i}_{j}$ - ⊕ recurrent of order one i.e. it is supposed to satisfies (2.3).

We differentiate (2.3) partially with respect to $\dot{x}^k$ and commute the equation thus obtained with respect to the indices j and k and thereafter use the commutation formula (1.3) along with the identities as have been given in (1.5), we get

$$(2.8)\quad {}^{+}R^{i}_{kj}{}^{+}|_{l} - p_l {}^{+}R^{i}_{kj} = - \dot{x}^h(\dot{\partial}_h {}^{+}R^{i}_{hj})^{+}|_{l} - {}^{+}R^{m}_{j}\Gamma^{i}_{kml} + {}^{+}R^{i}_{m}\Gamma^{m}_{kjl} + \Gamma^{m}_{kql}\dot{x}^q\dot{\partial}_m {}^{+}R^{i}_{j} + {}^{+}R^{i}_{j}\dot{\partial}_k p_l + p_l\dot{x}^h\dot{\partial}_k {}^{+}R^{i}_{hj}\ .$$

In a R^{+}- recurrentF_n^* of order one, we know that left hand side of (2.8) will vanish and accordingly, we get

$$(2.9)\quad \dot{x}^h(\dot{\partial}_k {}^{+}R^{i}_{hj})^{+}|_{l} = {}^{+}R^{i}_{m}\Gamma^{m}_{kjl} - {}^{+}R^{m}_{j}{}^{+}R^{i}_{kml} + \Gamma^{m}_{kql}\dot{x}^q\dot{\partial}_m {}^{+}R^{i}_{j} + {}^{+}R^{i}_{j}\dot{\partial}_k p_l + p_l\dot{x}^h\dot{\partial}_k {}^{+}R^{i}_{hj}\ .$$

Therefore, we can state:

THEOREM(2.4):

The necessary and sufficient condition in order that an $^{+}R_j^i$ - ⊕ recurrent F_n^* of order one be $^{+}R_{jk}^i$ - ⊕ recurrent F_n^* of order one is given by (2.9).

Using the commutation formula (1.3) and the set of identities as have been given in (1.5) in the result obtained after transvecting (2.8) by $\dot{x}^k$, we get

$$(2.10)\quad {}^{+}R_j^i\,{}^{+}|_l - p_l\,{}^{+}R_j^i = -\dot{x}^k\dot{x}^h(\dot{\partial}_k\,{}^{+}R_{hj}^i)\,{}^{+}|_l - {}^{+}R_j^m(\dot{\partial}_k\,\Gamma_{ml}^i)\dot{x}^k + {}^{+}R_m^i(\dot{\partial}_k\,\Gamma_{jl}^m)\dot{x}^k + (\dot{\partial}_m\,{}^{+}R_j^i)(\dot{\partial}_k\,{}^{+}R_j^i)(\dot{\partial}_k\Gamma_{ql}^m)\dot{x}^q\dot{x}^k + {}^{+}R_j^i\dot{x}^k(\dot{\partial}_k p_l) + p_l\dot{x}^h\dot{x}^k\dot{\partial}_k\,{}^{+}R_{hj}^i .$$

Using the homogeneity property of the connection parameter $\Gamma_{jk}^i(x,\dot{x})$ in (2.10), we get

$$(2.11)\quad {}^{+}R_j^i\,{}^{+}|_l - p_l\,{}^{+}R_j^i = -\dot{x}^k\dot{x}^h(\dot{\partial}_k\,{}^{+}R_{hj}^i)\,{}^{+}|_l + {}^{+}R_j^i\dot{x}^k\dot{\partial}_k\,p_l + p_l\dot{x}^h\dot{x}^k\dot{\partial}_k\,{}^{+}R_{hj}^i .$$

The left hand side of (2.11) will vanish automatically in a $^{+}R_j^i$ - ⊕ recurrent F_n^* of order one and accordingly, we get

$$(2.12)\quad \dot{x}^k\dot{x}^h(\dot{\partial}_k\,{}^{+}R_{hj}^i)\,{}^{+}|_l = {}^{+}R_j^i\dot{x}^k(\dot{\partial}_k\,p_l) + p_l\dot{x}^h\dot{x}^k\dot{\partial}_k\,{}^{+}R_{hj}^i .$$

THEOREM(2.4):

In a $^{+}R_j^i$ - ⊕ recurrent F_n^* of order one the identity as has been given by (2.12) is always true under the consideration that the connection parameter $\Gamma_{jk}^i(x,\dot{x})$ are homogeneous of degree zero in its directional arguments.

We now differentiate (2.3) ⊕ - covariantly with respect to $\dot{x}^m$ in the sense of (1.3) and then use (1.3) itself, we get

$$(2.13)\quad {}^{+}R_j^i\,{}^{+}|_{lm} = (p_l\,{}^{+}|_l + p_l\,p_m)\,{}^{+}R_j^i$$

We now allow a commutation in (2.13) with respect to the indices l and m and thereafter use the commutation formula (1.3) and the set of identities as have been given in (1.5) and get

$$(2.14)\quad -{}^{+}R_{lm}^r\dot{\partial}_r\,{}^{+}R_j^i + {}^{+}R_j^r\,{}^{+}R_{rlm}^i + N_{ml}^r({}^{+}R_j^i\,{}^{+}|_r) = (p_l\,{}^{+}|_m - p_m\,{}^{+}|_l)\,{}^{+}R_j^i .$$

Using (2.3) and (2.14) together, we get

$$(2.17)\quad (p_l\,{}^{+}|_m - p_m\,{}^{+}|_l + p_r\,N_{lm}^r)\,{}^{+}R_j^i = -{}^{+}R_{lm}^r\dot{\partial}_r\,{}^{+}R_j^i + {}^{+}R_j^r\,{}^{+}R_{rlm}^i - {}^{+}R_r^i\,{}^{+}R_{jlm}^r .$$

THEOREM(2.5):

The identity as has been given by (2.15) always holds in an $^{+}R_j^i$ - ⊕ recurrent F_n^* of order one.

The Bianchi identity in a Finsler space F_n^* equipped with semi-symmetric connection is given by

$$(2.16)\quad {}^{+}R_{ijk}^h\,{}^{+}|_l + {}^{+}R_{ikl}^h\,{}^{+}|_j + {}^{+}R_{ilj}^h\,{}^{+}|_k + {}^{+}R_{iljk}^h = 0 ,$$

where

$$(2.17)\quad {}^{+}R_{iljk}^h = {}^{+}R_{jk}^m\Gamma_{mil}^h + {}^{+}R_{kl}^m\Gamma_{mij}^h + {}^{+}R_{ij}^m\Gamma_{mjk}^h .$$

In (2.16) we use the process of ⊕ - covariant differentiation and get

(2.18) $^{+}R^{h}_{ijk}{}^{+}|_{l} = {}^{+}R^{h}_{ijk}{}^{+}|_{l} + {}^{+}R^{h}_{iqk}\Gamma^{q}_{jl} + {}^{+}R^{h}_{ijq}\Gamma^{q}_{kl}$.

In view of (2.1) the identity given by (2.18) can be rewritten in the following form

(2.19) $^{+}R^{h}_{ijk}{}^{+}|_{l} = p_{l}{}^{+}R^{h}_{ijk} + {}^{+}R^{h}_{iqk}\Gamma^{q}_{jl} + {}^{+}R^{h}_{ijq}\Gamma^{q}_{kl}$.

Allowing a cyclic interchange of the indices j, k, and l in (2.19), we shall get two more equations, adding (2.19) and the two equations thus obtained and thereafter using (2.18), we get

(2.20) $^{+}R^{h}_{ikl}{}^{+}|_{j} = p_{j}{}^{+}R^{h}_{ikl} + {}^{+}R^{h}_{iql}\Gamma^{q}_{kj} + {}^{+}R^{h}_{ikq}\Gamma^{q}_{lj}$

and (2.21) $^{+}R^{h}_{ilj}{}^{+}|_{k} = p_{k}{}^{+}R^{h}_{ilj} + {}^{+}R^{h}_{iqj}\Gamma^{q}_{lk} + {}^{+}R^{h}_{ilq}\Gamma^{q}_{jk}$.

Adding (2.19), (2.20) and (2.21) and use (2.16) thereafter the set of identities as have been given in (1.5), we get

(2.22) $p_{l}{}^{+}R^{h}_{ijk} + p_{j}{}^{+}R^{h}_{ikl} + p_{k}{}^{+}R^{h}_{ilj} + {}^{+}R^{h}_{iqk}N^{q}_{jl} + {}^{+}R^{h}_{iql}N^{q}_{kj} + {}^{+}R^{h}_{iqj}N^{q}_{lk} + E^{h}_{iljk} = 0$.

Transvection of (2.22) by $\dot{x}^{i}$ and thereafter use of identities given by (1.5), yields

(2.23) $3\,\lambda_{[l}{}^{+}R^{h}_{ijk]} + 3{}^{+}R^{h}_{p[k}N^{p}_{jl\,]} + \dot{x}^{i}E^{h}_{iljk} = 0$

where (2.24) $E^{h}_{iljk} \overset{\text{def}}{=} R^{m}_{jk}\Gamma^{h}_{mil} + R^{m}_{kl}\Gamma^{h}_{mij} + R^{m}_{lj}\Gamma^{h}_{mij}$.

Therefore, we can state:

THEOREM(2.6):

The Bianchi's identity assumes the form (2.23) in a R⁺- recurrent F^{*}_{n} of order one.

3. R - ⊕ RECURRENT F^{*}_{n} OF ORDER TWO.

DEFINITION(3.1):

A Finsler space F^{*}_{n} equipped with non-symmetric connection is said to R - ⊕ recurrent of order two if

(3.1) $^{+}R^{h}_{ijk}{}^{+}|_{lm} = b_{lm}{}^{+}R^{h}_{ijk}$, $^{+}R^{h}_{ijk} \neq 0$

where $b_{lm}(x\,,\dot{x})$ a covariant recurrence tensor field depending both upon positional and directional arguments. Allowing a commutation in (3.1) with respect to the indices l and m and thereafter using (1.3), we get

(3.2) $(\,b_{lm} - b_{lm}\,)\,{}^{+}R^{h}_{ijk} = -\,(\dot{\partial}_{q}{}^{+}R^{h}_{ijk}\,)^{+}R^{q}_{lm} + {}^{+}R^{q}_{ijk}{}^{+}R^{h}_{qlm} - {}^{+}R^{h}_{qjk}{}^{+}R^{q}_{ilm}$

$- {}^{+}R^{h}_{iqk}{}^{+}R^{q}_{jlm} - R^{h}_{ijq}{}^{+}R^{q}_{klm} + N^{q}_{ml}{}^{+}R^{h}_{ijk}{}^{+}|_{q}$.

As per the provisions of (3.2), we can therefore state :

THEOREM(3.1):

The recurrence tensor field is $b_{lm}(x\,,\dot{x})$ appearing in (3.1) is non- symmetric in a R⁺ - ⊕ recurrent F^{*}_{n} of order two.

We now differentiate (3.2), ⊕- covariantly with respect to x^{s} and then use (1.3) and (2.1) to get

(3.3) $^{+}R^{h}_{ijk}(\,b_{lm} - b_{ml}\,)\,{}^{+}|_{s} = p_{s}{}^{+}R^{h}_{ijk}\,(\,b_{lm} - b_{ml} - B_{ml}\,) + B_{ml}{}^{+}|_{s}{}^{+}R^{h}_{ijk}$

$+ \{{}^{+}R^{t}_{ijk}\Gamma^{h}_{qts} - {}^{+}R^{h}_{tjk}\Gamma^{t}_{qis} - {}^{+}R^{h}_{ink}\Gamma^{n}_{qjs} - {}^{+}R^{h}_{ijt}\Gamma^{t}_{qks}$

$- {}^{+}R^{h}_{tijk}\Gamma^{t}_{qrs}\,\dot{x}^{r}\,\}\,{}^{+}R^{q}_{lm}$,

where, we have written

(3.4) $B_{ml} \overset{\text{def}}{=} p_{t}\,N^{t}_{ml}$,

and also have taken into account the fact that the recurrence vector field p_{l} is independent of directional arguments.

We now allow a transvection of (3.3) with N^s_{rt} and get

$$(3.5)\ {}^{+}R^h_{ijk}(b_{lm} - b_{ml})^{+}|_s N^s_{rt} = B_{rt}\,{}^{+}R^h_{ijk}(b_{lm} - b_{ml} - B_{ml}) + {}^{+}R^h_{ijk} N^s_{rt} B_{ml}{}^{+}|_s + \{{}^{+}R^t_{ijk}\Gamma^h_{qts} - {}^{+}R^h_{tjk}\Gamma^t_{qis} - {}^{+}R^h_{ink}\Gamma^n_{qjs} - {}^{+}R^h_{ijt}\Gamma^t_{qks} - {}^{+}R^h_{tijk}\Gamma^t_{qrs}\dot{x}^r\}\,{}^{+}R^q_{lm} N^s_{rt}\ .$$

In accordance with the provisions as have been laid down in (3.5), we can therefore state:

THEOREM(3.2):

The relation (3.5) is always holds in a ⁺R - ⊕ recurrent F^*_n of order two.

We now differentiate (3.3), ⊕ - covariantly with respect to x^t and thereafter use (1.3) and get

$$(3.6)\ {}^{+}R^h_{ijk}(b_{lm} - b_{ml})^{+}|_{st} = b_{st}(b_{lm} - b_{ml} - B_{ml}){}^{+}R^h_{ijk} + p_s\,{}^{+}R^h_{ijk}(b_{lm} - b_{ml} - B_{ml})^{+}|_t + {}^{+}R^h_{ijk} B_{ml}{}^{+}|_{st} + p_t\{{}^{+}R^h_{ijk}(b_{lm} - b_{ml})^{+}|_s - {}^{+}R^h_{ijk} B_{ml}{}^{+}|_s\} - 2\,p_s p_t\,{}^{+}R^h_{ijk}(b_{lm} - b_{ml} - B_{ml}) + \{{}^{+}R^q_{ijk}\Gamma^h_{pqs}{}^{+}|_t - {}^{+}R^h_{qjk}\Gamma^q_{pis}{}^{+}|_t - {}^{+}R^h_{iqk}\Gamma^q_{pjs}{}^{+}|_t - {}^{+}R^h_{ijq}\Gamma^q_{pks}{}^{+}|_t - {}^{+}R^h_{qijk}\Gamma^q_{prs}{}^{+}|_t\dot{x}^r + ({}^{+}R^r_{ijk}\Gamma^h_{qrt} - {}^{+}R^h_{rjk}\Gamma^r_{qit} - {}^{+}R^h_{irk}\Gamma^r_{qjt} - {}^{+}R^h_{ijr}\Gamma^r_{qkt} - {}^{+}R^h_{rijk}\Gamma^r_{qst}\dot{x}^s)\Gamma^q_{prs}\dot{x}^r\}\,{}^{+}R^p_{lm}\ ,$$

where, we have taken into account the fact that recurrence vector p_l is independent of directional arguments along with the relations as have been given by (2.1), (3.3), (3.4).

We now transvect (3.6) successively by N^s_{bs} and N^t_{de} and thereafter use (3.6) and get

$$(3.7)\ 2\,{}^{+}R^h_{ijk}\,b_{[lm]}{}^{+}|_{st} N^s_{bc} N^t_{de} = (2\,b_{[lm]} - B_{ml})(b_{st} N^s_{bc} N^t_{de} - 2\,B_{bc} B_{de}){}^{+}R^h_{ijk} + {}^{+}R^h_{ijk}\{(2\,b_{[lm]} - B_{ml})^{+}|_t B_{bc} + B_{de}(2\,b_{[lm]} - B_{ml})^{+}|_t + B_{ml}{}^{+}|_{st} N^s_{bc} N^t_{de}\} + \{{}^{+}R^q_{ijk}\Gamma^h_{pqs}{}^{+}|_t - {}^{+}R^h_{qjk}\Gamma^q_{pis}{}^{+}|_t - {}^{+}R^h_{iqk}\Gamma^q_{pjs}{}^{+}|_t - {}^{+}R^h_{ijq}\Gamma^q_{pks}{}^{+}|_t - {}^{+}R^h_{qijk}\dot{x}^r\,\Gamma^q_{prs}{}^{+}|_t + ({}^{+}R^r_{ijk}\Gamma^h_{qrt} - {}^{+}R^h_{rjk}\Gamma^r_{qit} - {}^{+}R^h_{rik}\Gamma^r_{qjt} - {}^{+}R^h_{ijr}\Gamma^r_{qkt} - + {}^{+}R^h_{rijk}\Gamma^r_{qst}\dot{x}^s)\Gamma^q_{prs}\dot{x}^r\}\,{}^{+}R^p_{lm} N^s_{bc} N^t_{de}\ .$$

In accordance with the provisions of (3.7), we can therefore state:

THEOREM(3.3):

The recurrence tensor field $b_{lm}(x, \dot{x})$ satisfies (3.7) in a ⁺R - ⊕ recurrent F^*_n of order two.

For the curvature tensor ${}^{+}R^h_{ijk}$ The Bianchi identity in a recurrent F^*_n of order one has been given by (2.23). we differentiate (2.23), ⊕- covariantly with respect to x^m and then using (2.1), (3.1) and (2.23) to get

$$(3.8)\ (b_{lm} - p_l p_m){}^{+}R^h_{jk} + (b_{jm} - p_j p_m){}^{+}R^h_{kl} + (b_{km} - p_k p_m){}^{+}R^h_{ij} + {}^{+}R^h_{pk} \mathrm{N}^p_{jl}{}^{+}|_m - {}^{+}R^h_{pj} \mathrm{N}^p_{lk}{}^{+}|_m + {}^{+}R^h_{pl} \mathrm{N}^p_{kj}{}^{+}|_m + \dot{x}^i({}^{+}R^h_{iljk}{}^{+}|_m - p_m E^h_{iljk}) = 0$$

where we have written $b_{lm} = p_l{}^{+}|_m + p_l p_m$ and E^h_{iljk} has been given by (2.24) .

In accordance with (2.24), we can therefore state:

THEOREM(3.4):

The Bianchi's identity assumes the form as has been given by (3.8) in a R^+ - recurrent F_n^* of order two.

4. P_λ- FINSLER SPACES :

Let F_n be a Finsler space in which the (v) hv-torsion tensor P_{ijk} is given in the following form

(4.1) $P_{ijk} = \lambda C_{ijk} + b_i C_j C_k + b_j C_k C_i + b_k C_i C_j$,

where λ is a scalar function and b_i are the components of a covariant vector field , λ is homogenous of degree one in its directional arguments and b_i is homogenous of degree two in its directional arguments. Also $P_{ijo} = 0$ leads to $b_0 = 0$.

From (4.1), it can easily be verified that b_i is necessarily given by

(4.2) $b_i = = \frac{1}{C^2}[P_i - \frac{1}{3}(\lambda + \frac{2}{C^2} P_k C^k) C_i]$ for $C^2 \neq 0$.

From here we can easily tell that the form (4.1) depends upon scalar function λ and therefore we introduce the following definition.

DEFINITION (4.1):

A P_λ- Finsler space is a non-Riemannian Finsler space F_n(n ≥2) such that the (v) hv-torsion tensor of F_n is written in the form (4.1).

Before going into the details of this section we are giving the preliminary informations which shall be required for the later discussions in this sections. Let F_n be an n-dimensional Finsler space equipped with the fundamental function F$(x, \dot{x})$. The (v) hv- torsion tensor P_{ijk} and v-curvature tensor S_{hijk} are respectively given by the relations

(4.3) (a) $P^i_{hkm} = \frac{\partial \Gamma^{*i}_{hk}}{\partial \dot{x}^m} - C^i_{hm|k} - C^i_{hr} P^r_{km}$,

(b) $S^i_{hkm} = C^i_{rk} C^r_{hm} - C^i_{rm} C^r_{hk}$.

(c) $P_{hijk} = g_{im} P^m_{hjk}$, (d) $P_{ijk} = P_{hijk} \dot{x}^h$,

(e) $S_{hijk} = g_{im} S^m_{hjk}$.

Now, we propose to introduce some special Finsler spaces which shall be defined with the help of special forms of curvature and torsion tensors.

In a Finsler space it has been observed in general that $g_{ij(k)} = -2C_{ijk|o}$ where (k) stands for Berwald's process of covariant differentiation and o stands for transvection with $\dot{x}^i$.

A Finsler space in which $g_{ij(k)}= 0$ is called a Landsberg space, the characterization of such a space is given by

(4.4) $P_{ijk} = C_{ijk|o} = 0$.

The notion of P*- Finsler space has been introduced by Izumi [5].

A P*-Finsler space is a Finsler space F_n with the non-zero length C of the torsion vector C^i, if the (v)hv-torsion tensor of the space is written in the form:

(4.5) $P_{ijk}= \lambda\, C_{ijk}$

where $\lambda(x, \dot{x})$ is a scalar function which is given by

(4.6) (a) $\lambda = \frac{1}{C^2} P_i C^i$, (b) $C^i = g^{ij} C_j$, (c) $C_i= C^j_{ij}$,

(d) $P_i = P^j_{ij} = C_{i|o}$, (e) $C^2 = g^{ij}C_iC_j$.

A special Finsler space is called P-symmetric [9] if it be defined with a special form of hν-curvature tensor. The characterizing property of such a space is given by the relation

(4.7) P_{hijk} - P_{hikj}= 0.

A special Finsler space will be called P2-like [10] if in a non- Riemannian Finsler space F_n of dimension n (n>2) there exists a covariant vector field λ_i such that the hv-curvature tensor P_{ijkl} of F_n is written in the form

(4.8) $P_{ijkl} = \lambda_i C_{jkl} - \lambda_j C_{ikl}$.

In such a Finsler space the hν- curvature tensor P_{hijk} of F_n and the curvature tensor S_{hijk} of F_n vanishes.

An h-isotropic Finsler space is a Finsler space [1], [8] provided the h- curvature tensor R_{hijk} of the space is written in the form

(4.9) R_{hijk} = R ($g_{hj}g_{ik}$ - $g_{hk}g_{ij}$) ,

where R is a non-zero scalar function.

We give the following result (without proof) which will be found useful in the later discussions.

LEMMA (4.1):

An h-isotropic Finsler space satisfies P_{hijk} = P_{hikj} and S_{hijk}= 0 [1].

A C2- like Finsler space is a Finsler space F_n(n ≥2) with $C^2 \neq 0$ provided the (v) hv-torsion tensor C_{ijk} is written in the form

(4.10) $C_{ijk}=\frac{1}{C^2}C_iC_jC_k$

It is obvious from (4.10) that the necessary and sufficient condition in order that a non-Riemannian Finsler space F_n(n ≥2) be C2- like is that C_{ijk} be written in the form

(4.11) $C_{ijk} = L_iC_jC_k$+$L_jC_kC_i$+ $L_kC_iC_j$,

where $L_i = (\frac{1}{3C^2}C_i)$ are the components of a covariant vector field.

DEFINITION(4.2):

A semi P2- like space is a non-Riemannian Finsler space F_n(n ≥2) provided the (v) hv- torsion tensor P_{ijk} of F_n be written in the form

(4.12) $P_{ijk} = B_iC_jC_k + B_jC_kC_i + B_kC_iC_j$,

where B_i is an indicatory vector field positively homogeneous of degree two in its directional arguments.

From (4.12), it can easily be observed that the vector B_i can be given by

(4.13) $B_i = \frac{1}{C^2}(P_i - \frac{2}{3C^2}P_k\, C^k C_i)$, $C^2 \neq 0$.

OBSERVATION(4.1)

We have C_{ijk}= 0 in a Riemannian space which obviously implies C_i=0 and C^2=0 after taking into account (4.6c) and (4.6e) while conversely, if C^2=0, that is C_i=0, then according to Deicke's theorem such a space is necessarily Riemannian [4]. Thus from here we finally observe that the necessary and sufficient condition in order that a Finsler space be Riemannian is given by C^2=0.

In a two dimensional Finsler space F_2 [11], we have

(4.14) (a) $C_{ijk} = \frac{J}{L} m_i m_j m_k$,

(b) $C_i = \frac{J}{L} m_i$

(c) $P_{ijk} = \frac{J_{|o}}{L} m_i m_j m_k$.

where m_i is an unit vector orthogonal to the supporting element and J is the principal scalar. According to observation (4.1) we come to the conclusion that in a P_λ-Finsler space C^2 is non- vanishing. In case of two dimensions Finsler space P_{ijk} can be expressed in the form (4.1) which is an obvious consequence of (4.14). Therefore, we can state:

PROPOSITION (4.1):

A two dimensional non-Riemannian Finsler space F_2 is essentially a P_λ- Finsler space.

An obvious consequence of (4.1) is that a P_λ- Finsler space F_n will reduce into a P*- Finsler space when b_i vanishes identically while on the other hand a P_λ-Finsler space will reduce into a P2-like Finsler space if λ=0. From here we obvious that generalization of P* and semi-P2-like Finsler space is a P_λ-Finsler space. The relations (4.10) and (4.12) enable us to ensure that a C2-like Finsler space is a P_λ-Finsler space.

In case a P_λ-Finsler space becomes a Landsberg space then from (4.1) we get

(4.15) $C_{ijk} = -\frac{1}{\lambda}(b_i C_j C_k + b_j C_k C_i + b_k C_i C_j)$

and $b_i = -\frac{\lambda}{3C^2} C_i$.

Therefore, we can state the following:

THEOREM (4.1):

If a Finsler space is a Landsberg space then in order that it be a P_λ-Finsler space it is necessary and sufficient that it be a C2-like space.

Now, we propose to assume that a P_λ-Finsler space is semi-P2-like. Then when (4.1) and (4.12) taken together, we get

(4.16) $\lambda C_{ijk} = d_i C_j C_k + d_j C_k C_i + d_k C_i C_j$.

where $d_i = (B_i - b_i)$.

From (4.16) it is obvious that if d_i = 0 i.e. if B_i=b_i then λ=0, Otherwise in view of (4.10) the space will be C2-like. Therefore, we can state:

THEOREM (4.2):

The necessary and sufficient condition for a P_λ- Finsler space to be semi-P2-like is that the space is C2-like.

We now consider the case when a P_λ-Finsler space F_n becomes a P*-Finsler space with scalar function ρ. Then, when (4.1) and (4.5) taken together, we get

(4.17) $(\rho - \lambda) C_{ijk} = b_i C_j C_k + b_j C_k C_i + b_k C_i C_j$,

(4.17) clearly tells that the space is C2-like in case ρ≠λ while if ρ=λ, we get b_i=0 and have from (4.1) we find that the space is P*- Finsler space. Therefore, we can state:

THEOREM(4.3):

The necessary and sufficient condition in order that a P*- Finsler space be a P_λ-Finsler space is that it is a C2-like Finsler space .

THEOREM (4.4):

The necessary and sufficient condition in order that a P2-like Finsler space be P_λ-Finsler space is that it is C2-like.

The hv-curvature tensor P_{hijk} of a P_λ-Finsler space in view of (4.1) and (4.3) can be written in the following form

$$(4.18)\ P_{hijk} = \lambda\,|_h C_{ijk} - \lambda\,|_i C_{hjk} - \lambda\,|_h C_{ijk} - \lambda S_{hijk} + \Theta_{(hi)}\,[b_i\,|_h C_j C_k + C_j Q_{ikh} + C_k Q_{ijh} + C_i L'_{jkh} + L'_{ik} B_{jh} + C_i C_k F^i_{jh}]\ ,$$

where the quantities involved in (4.18) are given by

$$(4.19)\ (a)\ Q_{ikh} = B_i C_k\,|_h + C_i B_k\,|_h\ ,$$

$$(b)\ L_{jkh} = B_j C_k\,|_h + B_k C_j\,|_h\ ,$$

$$(c)\ L_{ik} = C_i B_k + B_i C_k\ ,$$

$$(d)\ B_{ij} = C_r C^r_{ij}\ , \qquad (e)\ F_{ij} = B_k C^k_{ij}\ .$$

The quantities appearing in (4.18) with dashes shall be obtained on replacing B_i and b_i in the corresponding quantities without dashes given by (4.19). b_i and B_{jh} are respectively given by (4.2) and (4.18d). The form (4.17) of P_{hijk} is more general than the forms of P_{hijk} as in P2-like and semi-P2-like Finsler spaces. We next consider a P_λ-and P- symmetric Finsler spaces. In accordance with (4.3) and (4.18), we get

$$(4.20)\ S_{hijk} = \frac{1}{2\lambda}\Theta_{(hi)}[L'_{ik}B_{jh} - L'_{ij}B_{kh} + C_i C_k F'_{jh} - C_i C_j F'_{kh}]\ ,$$

With the help of (4.20), we can therefore state the following:

THEOREM(4.5):

The necessary and sufficient condition in order that a P_λ-Finsler space be P- symmetric is that its v-curvature tensor S_{hijk} be written in the form as has been given in (4.20).

Next, we consider the case when a P_λ-Finsler space is h-isotropic. In such a case as per the provisions of lemma (4.1) such a space is P-symmetric and S_{hijk}=0.

Therefore, from (4.20) we get

$$(4.21)\ \Theta_{(hi)} = [L'_{ik}B_{jh} - L'_{ij}B_{kh} + C_i C_k F'_{jh} - C_i C_j F'_{kh}] = 0\ ,$$

Therefore, we can state the following:

THEOREM(4.6):

The relation (4.21) always holds in the case when a P_λ-Finsler space is h-isotropic.

We can rewrite (4.18) as under

$$(4.22)\ P_{hijk} = \lambda\,|_h C_{ijk} - \lambda\,|_i C_{hjk} + F_{hijk},$$

where (4.23) $F_{hijk} = \Theta_{(hi)}[b_i\,|_h C_j C_k + C_J Q'_{ikh} + C_k Q'_{ijh} + C_i L'_{jkh} + L'_{ik} B_{jh} + C_i C_k F^i_{jh}] - \lambda S_{hijk}$.

Using (4.8) and (4.22), we can therefore state:

THEOREM (4.7):

The necessary and sufficient condition in order that a P_λ-Finsler space be P2-like is that there should exist a covariant vector field F_h such that

$F_{hijk} = F_h C_{ijk} - F_i C_{hjk}$.

In the last we consider that case when a P_λ- Finsler space is admitting a concurrent vector field Y_i . Then with the help of the relations $P_{ijk}Y^i = 0$, $C_{ijk}Y^i = 0$ and (4.1), we get

(4.25) $b_i Y^i C_j C_k = 0$,

Contracting (4.24) by g^{jk} , we get

(4.26) $b_i Y^i = 0$.

Therefore, we can state:

THEOREM (4.8):

The vector b_i is orthogonal to Y_i in a P_λ-Finsler space admitting a concurrent vector field Y_i.

CONCLUSION

This paper has been devoted to the study of on some special Finsler spaces. The paper has been divided into four sections of which the first section is introductory, the second section deals with R⁺- recurrent F_n^* of order one. In this section we have derived results telling as to when a ${}^+R^i_{jk}$ - ⊕ recurrent F_n^* of order one will be R⁺ - ⊕ recurrent of order one, ${}^+R^i_j$ - ⊕ recurrent F_n^* of order one will be a ${}^+R^i_{jk}$ -⊕ recurrent of order one. In this section we have also derived the Bianchi's identity and few more identities which hold in a R⁺- recurrent F_n^* of order one. The third section deals with R⁺- recurrent F_n^* of order two. In this section we have observed that the recurrence tensor field $b_{lm}(x, \dot{x})$ is non-symmetric, few more relations and the Bianchi's identity have been derived in a R⁺- recurrent F_n^* of order two. In the fourth and the last section we have derived the conditions under which a Landsberg space in aP_λ-Finsler space, a P_λ- Finsler space is semi - P2-like, a P^*- Finsler space is a P_λ- Finsler space, a P_λ – Finsler space is P- symmetric, a P_λ - Finsler space is P2 like.

II. REFERENCES

[1]. Akbar-Zadeh, H. Les espaces de Finsler et certaines de leura generalizations, Ann. Sci. Ecole. Norn.Sup.(30) 80, 1-79,(1963).

[2]. Cartan, E. Les espaces de Finsler, Actualities, 79, Paris (1934).

[3]. Chaki, M.C. and Roy Chaudhary, A.N. On Ricci recurrent spaces of second order, J. Ind. Math. Soc. 2, 19, 279-287, (1967) .

[4]. Deicke, A. Uber die Finsler- Raume mit A_i= 0, Arch. Math.(4), 45- 51,(1953).

[5]. Izumi, H. On P^* - Finsler spaces. I, Memo Defense Academy, 16, 133-138(1976).

[6]. Kumar, A On a Q – recurrent Finsler space of the second order, Acta Ciencia Indica, 3 No. 2, 182-186, (1977) .

[7]. Matsumoto, M On Riemannian spaces with recurrent projective curvature, Tensor (N.S.), 19, 11-17, (1968) .

[8]. atsumoto, M. On h-isotropic and C^h - recurrent Finsler spaces, J.Math. Kyoto Univ. 11, 1-9,(1971).

[9]. Matsumoto, M. and Numata, S On semi C-reducible Finsler spaces with constant coefficients and C2- like Finsler spaces, Tensor(N.S.), 34, 218-222, (1980).

[10]. Matsumoto, M. On Finsler spaces with the curvature tensors P_{hijk} and S_{hijk} satisfying special conditions, Rep. on Math. Phys., 12, 77-87, (1977).

[11]. Rund, H. The Differential geometry of Finsler spaces, Springer-Verlag, Berlin(1959).

[12]. Vranceanu, C.E. Lectii de geometric differential,Vol I, EDP, BUC, (1962).

Cite this article as :

Ankit Maurya, K.B. Gupta, Jitendra Singh, "On Some Special Finsler Spaces", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 7 Issue 5, pp. 76-87, September-October 2020. Available at doi : https://doi.org/10.32628/IJSRST207524
Journal URL : http://ijsrst.com/IJSRST207524