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A New Class of Skew Normal Distribution : Tanh- Skew Normal Distribution and its Properties

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ABSTRACT

In this article we suggest a new class of skew normal distribution. It will be referred to as Tanh skew-normal distribution, where (Tanh) is a hyperbolic tangent function; a class of skew-normal distribution is proposed by considering a new skew function, It is not a probability distribution function, some properties of this new class distribution have been investigated. Several properties of this distribution have been discussed; parameters estimation using moments, moment generating function, maximum likelihood method, and Fisher information matrix are obtained. A numerical experiment was performed to see the behavior of MLEs. Finally, we apply this model to a real data-set to show that the new class distribution can produce a better fit than other classical Skew normal.

Keywords : Skew-Normal Distribution, Tanh Skew-Normal Distribution, Moments, Maximum Likelihood Estimators, Fisher Information Matrix.

I. INTRODUCTION

Azzalini, (1985) has presented the skew-normal (SN) distribution, it consists of modifying the normal probability density function by multiplication with a skewing function. Azzalini stated that the formal, $H(x) = 2\, f(x)\, \mathrm{G}(x)$, is the density of the skew distribution (pdf), where f is the density of a variable symmetric around (0), and (G),is the (cdf) of another independent random variable. By combining different symmetric distributions (normal, t, logistic, uniform, double exponential, etc.), numerous families of skewed distributions may be generated. This form of distribution is called a skew-symmetric, and it has been used in many applications to analyzes the asymmetric behavior of empirical data sets from various research fields, for more details about it, see (Ma and Genton, (2004)). In recent years, (Arellano-Valle et al. (2004) and G´omez et al. (2006)) introduced a class of skew-normal distributions, which related to the model (SN) introduced by Azzalini, most of those classes include the normal distribution as a particular case and satisfy similar properties as the normal family. (Hutson and Mudholkar (2000)), presented a normal a symmetric family of distributions with a different structure of the class (SN) considered by Azzalini, (1985), which is called epsilon skew-normal and it is denoted by (ESN(ε)), $\varepsilon > 0$ represents the asymmetry parameter, so that (ESN (0)) corresponds to the normal distribution. (Huang and Chen (2007), Chakraborty, Hazarika, and Ali (2012)) investigated the generalized skew symmetric distributions by introducing a skew function in place of cumulative distribution function (cdf), F(•), where a skew function G(•) is satisfying properties, 0≤G(x)≤1, G(x)+G(-x)=1, on the other hand, several of authors have provided alternative methods of Azallin, such as, (Elal-Olivero (2010), El-Damrawy et al, (2013b), Mahmoud, et al. (2015), Hazarika, Shah, and Chakraborty, (2019), Kundu and Gupta (2010).

This paper aims to introduce a new class of skew-normal distribution. It will be referred to as Tanh skew-normal distribution, where (Tanh) is a hyperbolic tangent function; a new class of skew-normal distribution is proposed by considering a new skew function where the skew function is not a cumulative distribution function (CDF). We expect that the proposed this model may be better (at least in terms of model fitting) than another classical skew-normal in certain practical situations.

The remainder of this paper is organized as follows: We discuss some of its basic properties are investigated, in Section 2. In Section 3, we provide expansions for a new skew-normal distribution, cumulative and density functions. In Section 4, we present various properties of the new model such as, moment generating function, and moments. In Section 5, the maximum likelihood estimator of the parameters of our model and an application to a real data set are obtained. Concluding remarks are presented in Section 6.

II. BASIC PROPERTIES

In this section we discuss the basic properties of the new distribution

2.1 . Define of The skew function:

The skew function $G(\lambda x)$ is using the hyperbolic tangent function (Tanh) and it's express by,

$$G(\lambda x) = 0.5\left(1 + Tanh\left[\frac{\lambda x}{2}\right]\right) \tag{1}$$

Where λ , is skew parameter, and the hyperbolic tangent function (Tanh) is the expression by

$$\text{Tanh}\left[\frac{\lambda x}{2}\right] = \frac{e^{\lambda x/2} - e^{-\lambda x/2}}{e^{\lambda x/2} + e^{-\lambda x/2}} \tag{2}$$

Substituting from (2) into (1) we get the skew function $G(\lambda x)$

$$G(\lambda x) = 0.5\left\{\frac{2}{1 + e^{-\lambda x}}\right\} = \frac{1}{1 + e^{-\lambda x}} = \left(1 + e^{-\lambda x}\right)^{-1} \tag{3}$$

By using the Taylor series expansion for $(1 + x)^{-1} = \sum_{k=0}^{\infty}(-1)^k x^k$,We get the skew function, $G(\lambda x)$ is,

$$G(\lambda x) = \begin{cases} \sum_{k=0}^{\infty}(-1)^k e^{-\lambda x}, & x \geq 0 \\ \sum_{k=0}^{\infty}(-1)^k e^{(k+1)\lambda x}, & x < 0 \end{cases} \tag{4}$$

Where , $\lambda > 0$

Similarly if $\lambda < 0$, the skew function is,

$$G(-\lambda x) = \begin{cases} \sum_{k=0}^{\infty}(-1)^k e^{(k+1)\lambda x}, & x > 0 \\ \sum_{k=0}^{\infty}(-1)^k e^{-\lambda x}, & x \leq 0 \end{cases} \tag{5}$$

2.2 . Properties of the skew function

a) The properties of the skew function are giving by,

$$G(\lambda x) + G(-\lambda x) = 1, 0 \leq G(\lambda x) \leq 1 \quad (6)$$

b) Plots of the skew function

The skew function of $G(\lambda x)$ for different choices of the parameter λ is plotted in Figure 1.

$$G(\lambda x) = 0.5(1 + Tanh[\lambda x/2])$$

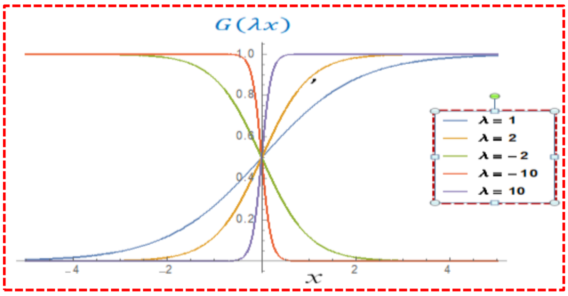

Figure 1.illustrates the shape of theskew function $(G(\lambda x)$for $\lambda = 1,\ 2, -2\ , 10, -10$

3. Expansions of The density and the cumulative distribution function

In this section, we introduce the probability density function and the cumulative distribution function of thestandardtanh skew- normal distribution.

3-1 **Expansions of The Probability density function**

The probability density function of thestandardtanh skew- normal distribution$(TSN(\lambda))$ is constructed using the formula ,

$$f(x, \lambda) = 2\phi(x)\, G(\lambda x), \quad -\infty < x < \infty \quad (7)$$

Where λ, a real number, is the skewness parameter,$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, is the probability density function of standard normal distribution and $G(\lambda x)(not\ cdf)$ is the skew function of definition its above equation (1,4). Substituting, $\phi(x), and\ G(\lambda x)$into formula (7), we get the probability density function,

$$f(x, \lambda) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\sum_{k=0}^{\infty} (-1)^k e^{-\lambda x}\right), & x > 0 \\ \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\sum_{k=0}^{\infty} (-1)^k e^{(k+1)\lambda x}\right), & x < 0 \end{cases} \quad (8)$$

Where,$\lambda > 0$

Similarly $\boldsymbol{\lambda} < 0$, the probability density function is

$$f(x, \lambda) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\sum_{k=0}^{\infty} (-1)^k e^{(k+1)\lambda x}\right), & x > 0 \\ \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\sum_{k=0}^{\infty} (-1)^k e^{-\lambda x}\right), & x < 0 \end{cases} \quad (9)$$

3.1.1 Properties of the probability density function

The properties of the density function of the standard tanh skew- normaldistribution$(TSN(\lambda))$are giving by,

a) $f(x,\lambda) \geq 0$
b) $\int_{-\infty}^{\infty} f(x,\lambda)\, dx = 1$
c) if $\lambda = 0$, then we get the standard normal distribution and is given, $f(x,\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
d) If $X \sim TSN(X,\lambda)$ then $-X \sim TSN(X,-\lambda)$
e) Plots of the probability density function : the plots of the probability density function of $TSN(\lambda)$ distribution for different choices of the parameter λ, is plotted in Figure 2.

$$f(x,\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(1 + \mathrm{Tanh}\left[\frac{\lambda x}{2}\right]\right), -\infty < x < \infty, \quad \lambda \in R$$

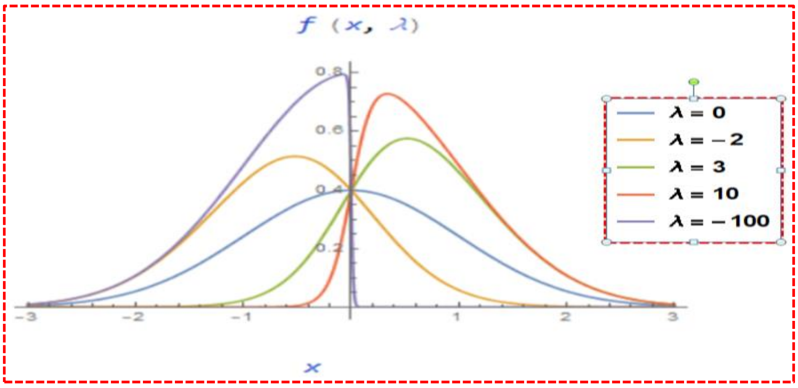

Figure2. illustrates the shape of the probability density function for λ= 0, -2, 3, 10, -100.

Remark 1.

from the Figure2 , when ,$\lambda \to \pm\infty$ the probability density function of the standard Tanh skew- normal distribution(*TSN(λ))* converges to the half normal density function.

Remark 2.

Throughout the rest of this paper (unless otherwise stated), we shall assume that $\lambda > 0$, since the corresponding results for $\lambda < 0$, can be obtained using the fact that *(–X*) has the density function

$$f(x) = 2\phi(x)\, G(-\lambda x).$$

3.2 Expansions of the cumulative distribution function:

In this subsection, we introduce the expansion forms for thecumulative distribution function(cdf) for the standardTanhskew normal distribution , we write it's at following ,

$$F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_{0}^{x} f(x)\, dx + \int_{-\infty}^{0} f(x)\, dx$$

where $F(x)$ is definite the cumulative function.

1) If $x \geq 0$, $\lambda > 0$, and writing , $F_1(x) = A + B$

$$A = \int_{0}^{x} f(x)\, dx, \qquad B = \int_{-\infty}^{0} f(x)\, dx$$

Substituting about $f(x)$ in equation (8), we getting,

$$A = \int_{0}^{x} \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\sum_{k=0}^{\infty} (-1)^k e^{-k\lambda x}\right) dx = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} (-1)^k \int_{0}^{x} e^{-\frac{x^2}{2} - k\lambda x}\, dx\,,$$

$$A = \sum_{k=0}^{\infty} (-1)^k \, e^{\left(\frac{k^2\lambda^2}{2}\right)} \left\{ erf\left[\frac{1}{\sqrt{2}} \; (x + k\lambda)\right] - erf\left[\frac{1}{\sqrt{2}} \; (k\lambda)\right]\right\}$$

Similarly, $B = \int_{-\infty}^{0} f(x)\,dx B = \int_{-\infty}^{0} f(x)\,dx = \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \int_{-\infty}^{0} e^{-\frac{1}{2}x^2 + (k+1)\lambda x} dx$

$$B = \sum_{k=0}^{\infty} (-1)^k \, e^{\frac{(k+1)^2\lambda^2}{2}} \left[erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right]\right]$$

From above equations (A,B), we get the cumulative function, $\lambda > 0$

$$F_1(x) = \begin{cases} \sum_{k=0}^{\infty} (-1)^k \, e^{\left(\frac{k^2\lambda^2}{2}\right)} \left\{ erf\left[\frac{x + k\lambda}{\sqrt{2}}\right] - erf\left[\frac{k\lambda}{\sqrt{2}}\right]\right\} \\ + \sum_{k=0}^{\infty} (-1)^k \, e^{\frac{(k+1)^2\lambda^2}{2}} erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right], \qquad x \geq 0, \end{cases} \tag{10}$$

2) If $\boldsymbol{x} < 0, \;\; \lambda > 0,$ the cumulative distribution function is,

$$F_2(x) \;\; = \int_{-\infty}^{x} f(x)\,dx == \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \int_{-\infty}^{x} e^{\frac{-x^2}{2} + (k+1)\lambda x} \, dx$$

$$F_2(x) = \sum_{k=0}^{\infty} (-1)^k \, e^{\frac{(k+1)^2\lambda^2}{2}} \left\{ erfc\left[\frac{\lambda(k+1) \; - \; x}{\sqrt{2}}\right]\right\}, x < 0 \tag{11}$$

Since, from equations (10),(11), the cumulative distribution function is, $F(x) =$

$$\boldsymbol{F_1}(x) + \boldsymbol{F_2}(x) = \begin{cases} \sum_{k=0}^{\infty} (-1)^k \, e^{\left(\frac{k^2\lambda^2}{2}\right)} \left\{ erf\left[\frac{(x+k\lambda)}{\sqrt{2}}\right] - erf\left[\frac{(k\lambda)}{\sqrt{2}}\right]\right\} \\ +\sum_{k=0}^{\infty} (-1)^k \, e^{\frac{(k+1)^2\lambda^2}{2}} erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right], \qquad x \geq 0 \\ \sum_{k=0}^{\infty} (-1)^k \, e^{\frac{(k+1)^2\lambda^2}{2}} erfc\left[\frac{(k+1)\lambda - x}{\sqrt{2}}\right], \; x < 0 \end{cases} \tag{12}$$

Similarly, if $\lambda < 0$, By using the dinsty function in equation (9) we getting,

The Cumulative distribution function is,

$$F(x) = \begin{cases} \sum_{k=0}^{\infty} (-1)^k \, e^{\frac{(k+1)^2\lambda^2}{2}} \left\{ erf\left[\frac{1}{\sqrt{2}} \; x - \frac{(k+1)\lambda}{\sqrt{2}}\right] + erf\left[\frac{(k+1)\lambda}{\sqrt{2}}\right]\right\} \\ + \sum_{k=0}^{\infty} (-1)^k e^{\left(\frac{k^2\lambda^2}{2}\right)} \left\{1 + erf\left[\frac{k\lambda}{\sqrt{2}}\right]\right\}, \qquad x \geq 0 \\ \sum_{k=0}^{\infty} (-1)^k \, e^{\left(\frac{k^2\lambda^2}{2}\right)} \left\{1 + erf\left[\frac{1}{\sqrt{2}} \; x + \frac{k\lambda}{\sqrt{2}}\right]\right\}, \qquad x < 0 \end{cases} \tag{13}$$

where (*erf*) is denoted error function, and its donation, (1) $erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}$, (erfc) is denoted complement error function, (2) $erfc(x) = 1 - erf\,(x)$)

3..2.1 plots of the cumulative distribution function

the plots of The cumulative distribution function with different choices parameter λ, is shown in figure 4.

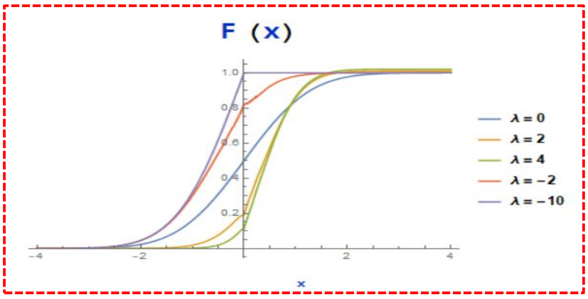

Figure 3. Plots of the cumulative distribution function (cdf) of *TSN(λ))* ,λ= 0, 2, 4, -2, -10.

3.2.2. Properties of the Cumulative distribution function

a) if $\lambda = 0$, The Cumulative distribution function is stander normal distribution

b) The skewness of the distribution($TSN(\lambda)$) increases as the value of λ increases in absolute.

c) When $\lambda \to \pm\infty$ The Cumulative distribution function of the standard tanh skew- normal distribution (*TSN (λ))* converges to half normal density function.

d) $0 \leq F(x) \leq 1$

4. Statistical Properties

In this section, Let us view different moments of $TSN(0,1,\lambda)$distribution. By using the moment , we can study some of the most important characteristics and features of a distribution, such as moment generating function, characteristic function and moments.

4.1. Moment Generating Function

If X has the $TSN(\lambda)$distribution, *and* $\lambda > 0$ then the mgf is:

$$M_x(t) = \sum_{k=0}^{\infty}(-1)^k\left(e^{\frac{1}{2}(k\lambda-t)^2}\left[erfc\left(\frac{1}{\sqrt{2}}(k\lambda - t)\right)\right] + e^{\frac{1}{2}((k+1)\lambda+t)^2}\left[erfc\left(\frac{1}{\sqrt{2}}((k+1)\lambda+t)\right)\right]\right) \quad (14)$$

4.2. The characteristic function

The characteristic function of X is:

$$E[\exp(itX)] = \sum_{k=0}^{\infty}(-1)^k\left\{e^{\frac{[(k+1)\lambda+it]^2}{2}}\text{erfc}\left[\frac{(k+1)\lambda+it}{\sqrt{2}}\right] - e^{\frac{1}{2}(k\lambda-it)^2}\left(\text{erfc}\left[\frac{k\lambda-it}{\sqrt{2}}\right]\right)\right\} \quad (15)$$

Where , $\lambda > 0, i = \sqrt{-1}$

4.3 General Moments

If $X \tilde{} TSN(0,\lambda)$ is a random variable, the r-th moments of *X, is defined as*

$$\acute{\mu}_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx,$$

Using these representations and properties in Sections 2.3 of (Prudnikov **et** *al.*[**13,** volume 1]), one can obtain,

$$E(X^r) = \lambda\sum_{k=0}^{\infty}(-1)^k\left\{\frac{\partial^r}{\partial q^r}e^{\frac{(k+1)^2\lambda^2}{2}}erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right] + (-1)^r\frac{\partial^r}{\partial s^r}e^{\frac{k^2\lambda^2}{2}}erfc\left[\frac{k\lambda}{\sqrt{2}}\right]\right\} \quad (16)$$

Where $q = \lambda(k+1),\ s = \lambda k$

4.2.1 Mean

For r= 1 Equation (16), yields the mean $(E(X) = \acute{\mu}_1)$ of $TSN(\lambda)$that is given by:

$$\acute{\mu}_1 = E(X) = \sum_{k=0}^{\infty} (-1)^k \left\{ \lambda(k+1)\, e^{\frac{(k+1)^2\lambda^2}{2}} \mathrm{erfc}\left[\frac{(k+1)\lambda}{\sqrt{2}}\right] - k\lambda e^{\frac{k^2\lambda^2}{2}} \left(\mathrm{erfc}\left[\frac{k\lambda}{\sqrt{2}}\right]\right) \right\} \qquad (17)$$

We derive the moment generating function at $t = 0$ we can getting the four first moments by using mgf$\left(M_x(t)\right)$, and formula $\acute{\mu}_r$ is

$$\acute{\mu}_r = \frac{d^r}{dt^r}\left(M_x(t)\right), \qquad t = 0$$

$$2)\ E(X^2) = \sum_{k=0}^{\infty} (-1)^k \left\{ \begin{array}{c} -\sqrt{\frac{2}{\pi}} k\lambda + (k^2\lambda^2 + 1) e^{\frac{k^2\lambda^2}{2}} erfc\left[\frac{k\lambda}{\sqrt{2}}\right] - \sqrt{\frac{2}{\pi}} \lambda(k+1) \\ +((k+1)^2\lambda^2 + 1) e^{\frac{(k+1)^2\lambda^2}{2}} \left(erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right]\right) \end{array} \right\} \qquad (18)$$

$$3)\ E(X^3) = \sum_{k=0}^{\infty} (-1)^k \left\{ \begin{array}{c} -\sqrt{\frac{2}{\pi}} (2k+1)\lambda^2 - k\lambda(3 + k^2\lambda^2) e^{0.5k^2\lambda^2} erfc\left[\frac{1}{\sqrt{2}}\, k\lambda\right] \\ + \lambda(k+1)((k+1)^2\lambda^2 + 3)\, e^{\frac{1}{2}(k+1)^2\lambda^2} erfc\left[\frac{1}{\sqrt{2}}((k+1)\lambda)\right] \end{array} \right\} \qquad (19)$$

$$4)\ E(X^4) = \sum_{k=0}^{\infty} (-1)^k \left\{ \begin{array}{c} -\sqrt{\frac{2}{\pi}} \lambda(k+1)(5 + (k+1)^2\lambda^2) - \sqrt{\frac{2}{\pi}} k\lambda(5 + k^2\lambda^2) \\ +(k^4\lambda^4 + 6k^2\lambda^2 + 3) e^{\frac{k^2\lambda^2}{2}} erfc\left[\frac{k\lambda}{\sqrt{2}}\right] \\ +((k+1)^4\lambda^4 + 6(k+1)^2\lambda^2 + 3) e^{\frac{(k+1)^2\lambda^2}{2}} erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right] \end{array} \right\} \qquad (20)$$

4.2.2 Central Moments

The central moments (μ_r) of X can be calculated as,

$$\mu_r = E[(X - \acute{\mu}_1)^r] \quad , \text{ where } \acute{\mu}_1 \text{ is the mean of } X,$$

$$\mu_r = \int_{-\infty}^{\infty} (x - \acute{\mu}_1)^r f(x)\, dx, \qquad \lambda > 0$$

By using binomial expansion

$$(a-b)^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} b^j a^{m-j}$$

$$\mu_r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} (\acute{\mu}_1)^j \int_{-\infty}^{\infty} x^{r-j} f(x)\, dx, \qquad \lambda > 0$$

$$\mu_r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} (\acute{\mu}_1)^j \acute{\mu}_{r-j} \qquad (21)$$

Where $\mu_1 = \acute{\mu}_1$.Then, the variance of $TSN(\lambda)$ distribution is given by:$\mu_2 = \acute{\mu}_2 - \acute{\mu}_1^2$, $\mu_3 = \acute{\mu}_3 - 3\,\acute{\mu}_2\,\acute{\mu}_1 + 2\,\acute{\mu}_{1,}^2$, $\mu_4 = \acute{\mu}_4 - 4\,\acute{\mu}_3 + 6\,\acute{\mu}_2\,\acute{\mu}_1^2 - 3\,\acute{\mu}_{1,}^4$, etc. Also, the skewness$\gamma_1 = \mu_3 / \mu_2^{\frac{3}{2}}$, and kurtosis $\gamma_2 = \mu_4 / \mu_2^2$ follow from the second, third and fourth moments.

4.2.3 Mean Deviation

Let X be a random variable that follows $TSN(\lambda)$ distribution with median m and mean μ. In this subsection, we inferred the mean deviation from the mean and the median.

1) The form of the mean deviation from the mean of the $TSN(\lambda)$ distribution is,

$$E(|X-\mu|) = \int_{-\infty}^{\infty} |X-\mu|\, f(x)\, dx,$$

$$= \int_{-\infty}^{\mu} (\mu - x)\, f(x)\, dx \quad + \quad \int_{\mu}^{\infty} (x-\mu)\, f(x)\, dx$$

$$E(|X-\mu|) = \; 2\,\mu\, F(\mu) \;\; -2\int_{-\infty}^{\mu} x f(x)\, dx\,,$$

- If$\mu \geq 0$

$$E(|X-\mu|) = \; 2\,\mu \left\{ \begin{array}{l} \sum_{k=0}^{\infty} (-1)^k \; e^{\left(\frac{k^2\lambda^2}{2}\right)} \left\{ erf\left[\frac{\mu + k\lambda}{\sqrt{2}}\right] - erf\left[\frac{k\lambda}{\sqrt{2}}\right] \right\} \\ + \sum_{k=0}^{\infty} (-1)^k \; e^{\frac{(k+1)^2\lambda^2}{2}} \, erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right], \end{array} \right\}$$

$$-2\sum_{k=0}^{\infty} (-1)^k \left\{ (k+1)\lambda\, e^{\frac{(k+1)^2\lambda^2}{2}} erfc\left[\frac{(k+1)\lambda}{\sqrt{2}}\right] - 1 \right\} - 2\int_{0}^{\mu} x f(x)\, dx\,,$$

- If $\mu < 0$

$$E(|X-\mu|) \;\; = \;\; 2\,\mu \sum_{k=0}^{\infty} (-1)^k \; e^{\frac{(k+1)^2\lambda^2}{2}} er\, fc\left[\frac{(k+1)\lambda \; - \; \mu}{\sqrt{2}}\right] - 2\int_{-\infty}^{\mu} x f(x)\, dx\,,$$

3) The form of the mean deviation from the median of the $TSN(\lambda)$ distribution

$$E(|X-m|) = \int_{-\infty}^{\infty} |X-m|\, f(x)\, dx,$$

The expression for $E(|X-m|)$is the same with μ replaced by m.

4.4 Location Scale Extension:

The location and scale extension of $TSN(\mu,\sigma,\lambda)$distribution is as follows. If Z ~ $TSN(\mu,\sigma,\lambda)$ then $X = Z\,\sigma + \mu$, is said to be the location (μ) and scale (σ) extension of Z and has the density function is given by

$$f(z,\lambda) = 2\phi\left(\frac{x-\mu}{\sigma}\right) G\left(\frac{\lambda(x-\mu)}{\sigma}\right),$$

where, (z, μ,σ,λ) εR, and $\sigma > 0$ We denote it by $X\sim\; TSN(\mu,\sigma,\lambda)$

5. Parameter Estimation and Applications

5.1 Likelihood function and maximum likelihood estimates.

In this section, the ML method is considered to estimate the parameters of $TSN(\mu,\sigma,\lambda)$ distribution. Let $(x_1, x_2, \ldots .. x_n)$ be a random sample with size n from the $TSN(\mu,\sigma,\lambda)$, with *pdf*by $f(x_i,\mu,\sigma,\lambda) = 2\phi\left(\frac{x_i-\mu}{\sigma}\right) G\left(\frac{\lambda(x_i-\mu)}{\sigma}\right)$, Also, we assume that$\Theta = (\mu,\sigma,\lambda)^{\tau}$is the $(r * 1)$unknown parameter vectors, the log-likelihood function is defined by:

$$l(\Theta) = \text{Log}\; \text{L}_n(\Theta/x_1, x_2, \ldots., x_n) = \text{Log}\; \text{L}_n(\Theta/x_i))$$

Where, $\text{L}_n(\Theta/x_i) = \prod_{i=1}^{n} f(x_i,\Theta)$ is the likelihood function.We can derive the likelihood function of $TSN(\mu,\sigma,\lambda)$ distribution as,

$$\text{L}_n(\Theta/x_i) = \prod_{i=1}^{n} 2\phi\left(\frac{x_i-\mu}{\sigma}\right) G\left(\frac{\lambda(x_i-\mu)}{\sigma}\right)$$

Where,$\phi\left(\frac{x_i-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}, ,-\infty < \; x < \infty, \;\; \mu\, \epsilon\, R\,, \;\; \sigma > 0$

is the probability density function of normal distribution and $G(\lambda x)$ is the skew function

$$G(\lambda x) = 0.5\left(1 + \text{Tanh}\left[\frac{\lambda x}{2}\right]\right) = \frac{1}{1 + e^{-\lambda\left(\frac{x_i-\mu}{\sigma}\right)}}\mu \epsilon R, \quad \sigma > 0, \ \lambda \epsilon R$$

Where, λ, is skew parameter

$$L_n(\mu, \sigma, \lambda/x_i) = \prod_{i=1}^{n} \frac{2}{\sigma\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}}{\left[1 + e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}$$

Thus, the log-likelihood function of $TSN(\mu, \sigma, \lambda)$ distribution is obtained as:

$$l(\Theta) = \text{Log } L_n(\mu, \sigma, \lambda | x_i) = \text{Log }\left[\frac{2}{\sigma\sqrt{2\pi}}\right]^n - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 - \sum_{i=1}^{n}\text{Log}\left[1 + e^{-\frac{\lambda(x_i - \mu)}{\sigma}}\right] \quad (22)$$

$$l(\Theta) = \text{n Log } (2) - \text{n Log } \sigma - \frac{n}{2}Log\ (2\pi) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 - \sum_{i=1}^{n}\text{Log }\left[1 + e^{-\frac{\lambda(x_i - \mu)}{\sigma}}\right] \quad (23)$$

The first derivatives of Equation (23) with respect to μ, σ,and λ respectively is given by:

$$\frac{\partial l(\Theta)}{\partial \mu} = \frac{\partial l(\mu, \sigma, \lambda)}{\partial \mu} = \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right) - \frac{\lambda}{\sigma} \quad (24)$$

$$\frac{\partial l(\Theta)}{\partial \sigma} = \frac{\partial l(\mu, \sigma, \lambda)}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 - \frac{\lambda}{\sigma}\sum_{i=1}^{n}\frac{\frac{(x_i-\mu)}{\sigma}e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1 + e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]} \quad (25)$$

$$\frac{\partial l(\Theta)}{\partial \lambda} = \frac{\partial l(\mu, \sigma, \lambda)}{\partial \lambda} = \sum_{i=1}^{n}\frac{\frac{(x_i-\mu)}{\sigma}e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1 + e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]} \quad (26)$$

Equate the Equations (24)-(26) to zero and solving them simultaneously yield the maximum likelihood estimators (MLEs) of $TSN(\mu, \sigma, \lambda)$ distribution parameters, then, from (24, 25, and 26), we get

$$\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right) = \lambda\sum_{i=1}^{n}\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1 + e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]} \quad (27)$$

$$\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 - \lambda\sum_{i=1}^{n}\frac{\frac{(x_i-\mu)}{\sigma}e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1 + e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]} = n \quad (28)$$

$$\sum_{i=1}^{n}\frac{\frac{(x_i-\mu)}{\sigma}e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1 + e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]} = 0 \quad (29)$$

Clearly, these equations are not in explicit form, the solutions can be found by using a numerical method such as the Newton-Raphson procedure to obtain the MLEs of the parameters μ, σ,and λ To obtain the asymptotic confidence intervals (CIs) for the parameters of the $TSN(\mu, \sigma, \lambda)$ distribution, the $3 \times 3\ I_n(\Theta) = I(\mu, \sigma, \lambda)$is required. Under certain regularity conditions, the MLEs asymptotically have multivariate normal distribution with mean vector ($\underset{\smile}{\Theta} = 0{,}0{,}0$) and variance- covariance matrix, which is given by the inverse of Fisher information:

$$I_n^{-1}(\widehat{\Theta}) = I_n^{-1}(\hat{\mu}, \hat{\sigma}, \hat{\lambda})(30)$$

for more details about asymptotic confidence intervals. The $I_n(\Theta)$depends on Θ, the observed Fisher information matrix $I_n(\widehat{\Theta})$May be used instead of the $I_n(\Theta)$ the estimation of the variance of

$$I_n(\widehat{\Theta}) = I_n(\hat{\mu}, \hat{\sigma}, \hat{\lambda})$$

$$I_n(\widehat{\Theta}) = -\begin{bmatrix} I_{\mu\mu} & I_{\mu\sigma} & I_{\mu\lambda} \\ I_{\sigma\mu} & I_{\sigma\sigma} & I_{\sigma\lambda} \\ I_{\lambda\mu} & I_{\lambda\sigma} & I_{\lambda\lambda} \end{bmatrix}_{(\hat{\mu}=\mu,\hat{\sigma}=\sigma,\hat{\lambda}=\lambda)} \quad (31)$$

Where, $I_{ij} = E\left[\frac{\partial^2 Log\,\ell}{\partial\theta_i\partial\theta_j}\right]$regrettably, the accurate mathematical expressionsfor the above expectation are very hard to obtain. Therefore, the observed Fisher information matrix is given by which is obtained by$I_{ij} = \left[\frac{\partial^2 Log\,\ell}{\partial\theta_i\partial\theta_j}\right]$ dropping the expectation on operation The elements of the $I_n(\widehat{\Theta})$are given by the following equations,

$$I_{\mu\mu} = E\left[\frac{\partial^2}{\partial\mu^2}\operatorname{Log} L_n(\mu,\sigma,\lambda)\right] = \frac{-n}{\sigma^2} - \frac{\lambda^2}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right) + \frac{\lambda^2}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right)^2 \quad (32)$$

$$I_{\sigma\sigma} = \begin{pmatrix} \frac{n}{\sigma^2} - \frac{2}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{x_i-\mu}{\sigma}\right)^2 + \frac{2\lambda}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{\frac{(x_i-\mu)}{\sigma} e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right) - \frac{\lambda^2}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{\left(\frac{x_i-\mu}{\sigma}\right)^2 e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right) \\ + \frac{\lambda^2}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{\left(\frac{x_i-\mu}{\sigma}\right) e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right)^2 \end{pmatrix} \quad (33)$$

$$I_{\lambda\lambda} = -\sum_{i=1}^{n} E\left(\frac{\left(\frac{x_i-\mu}{\sigma}\right)^2 e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right) + \sum_{i=1}^{n} E\left(\left(\frac{x_i-\mu}{\sigma}\right)^2 \left(\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right)^2\right) \quad (34)$$

$$\begin{aligned} I_{\mu\sigma} = I_{\sigma\mu} &= E\left[\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial\sigma}\operatorname{Log} L_n(\mu,\sigma,\lambda)\right)\right] \\ &= -\frac{2}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{x_i-\mu}{\sigma}\right) + \frac{\lambda}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}\right) - \frac{\lambda^2}{\sigma^2}\sum_{i=1}^{n} E\left(\frac{\frac{(x_i-\mu)}{\sigma} e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}\right) \\ &+ \frac{\lambda^2}{\sigma^2}\sum_{i=1}^{n} E\left(\left(\frac{x_i-\mu}{\sigma}\right)\left(\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}\right)^2\right) \end{aligned} \quad (35)$$

$$\begin{aligned} I_{\mu\lambda} = I_{\lambda\mu} &= E\left[\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial\lambda}\operatorname{Log} L_n(\mu,\sigma,\lambda)\right)\right] \\ &= \frac{-1}{\sigma}\sum_{i=1}^{n} E\left(\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}\right) \\ &+ \frac{\lambda}{\sigma}\sum_{i=1}^{n} E\left(\frac{\frac{(x_i-\mu)}{\sigma} e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}\right) + \frac{\lambda}{\sigma}\sum_{i=1}^{n} E\left(\left(\frac{x_i-\mu}{\sigma}\right)\left(\frac{e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}\right)^2\right) \end{aligned} \quad (36)$$

$$\boldsymbol{I_{\sigma\lambda}} = \boldsymbol{I_{\lambda\sigma}} = E\left[\frac{\partial}{\partial\sigma}\left(\frac{\partial}{\partial\lambda}\mathrm{Log}\, L_n(\mu,\sigma,\lambda)\right)\right]$$
$$= \frac{-1}{\sigma}\sum_{i=1}^{n} E\left\{\frac{\frac{(x_i-\mu)}{\sigma} e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right\} + \frac{\lambda}{\sigma}\sum_{i=1}^{n} E\left\{\frac{\left(\frac{x_i-\mu}{\sigma}\right)^2 e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right\}$$
$$+\frac{\lambda}{\sigma}\sum_{i=1}^{n} E\left(\frac{\left(\frac{x_i-\mu}{\sigma}\right) e^{-\frac{\lambda(x_i-\mu)}{\sigma}}}{\left[1+e^{-\frac{\lambda(x_i-\mu)}{\sigma}}\right]}\right)^2 \quad (37)$$

The approximate (1–δ)100% CIs of the parameters of $TSN(\mu,\sigma,\lambda)$ are respectively, given by: $\hat{\mu} \pm Z_{\frac{\delta}{2}}\sqrt{V(\hat{\mu})}$, σ^ ± $Z_{\frac{\delta}{2}}\sqrt{V\,(\,\sigma\hat{}\,)}$, and λ^ ± $Z_{\frac{\delta}{2}}\sqrt{V\,(\lambda\hat{}\,)\,}$ where, $V(\hat{\mu})$, $V\,(\sigma\hat{}\ )$) and $V\,(\lambda\hat{}\ )$are the variances of$\hat{\mu}$, σ^ and λ^ which are given by the diagonal elements of $I_n{}^{-1}(\widehat{\Theta}) = I_n{}^{-1}(\hat{\mu},\hat{\sigma},\hat{\lambda})$and $Z_{\frac{\delta}{2}}$ is the upper(δ / 2) percentile of the standard normal distribution.

5.2. Real Life Applications

In this section, we illustrate an application of the TSN(μ,σ,λ) distribution on the skew data set, for example, the data set is the white cells count (WCC) of 202 Australian athletes, given in (Cook and Weisberg (1994)) for the data fitting. Using GenSA package in R we have fitted our proposed distribution. We apply the values of log likelihood function (Log), Kolmogorov-Smirnov (K-S), P-value of (K-S) statistics to verify which distribution better fits these data. The model selection was carried out using the AIC (Akaike information criterion), the Second Order of Akaike Information Criterion (AICc) and the BIC (Bayesian information criterion)

$$AIC = -2 * \ l(\widehat{\Theta}) + 2k$$
$$AICc = AIC + \frac{2k(k+1)}{n-k-1}$$
$$BIC = -2 * \ l(\widehat{\Theta}) + k * Log(n)$$

where,(Θ ^) denotes the log-likelihood function evaluated at the maximum likelihood estimates, (k) is the number of parameters, and (n) is the sample size.

We compare the results of our distribution TSN(μ,σ,λ), with the corresponding distribution of the skew-normal SN(μ,σ,λ) distribution Azzalini (1985),Skew logistic SL(μ,σ,λ) ,Alpha Skew-Normal Distribution $ASN(\mu,\sigma,\alpha)$ and normal N(μ,σ^2) distribution, MLEs, AIC, BIC, and AICc for the parameters of distributions are given in Table 1.

Table 1: MLE's, , AIC, BIC and AICc for the real data set.

Model	MLE estimate				Statistics		
	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\lambda}$	α	AIC	BIC	AICc
$\boldsymbol{N(\mu,\sigma^2)}$	7.109	1.796	-	-	813.838	820.455	724.245
$\boldsymbol{SN(\mu,\sigma,\lambda)}$	5.105	2.691	1.729	-	798.322	808.247	784.587
$\boldsymbol{SL(\mu,\sigma,\lambda)}$	5.319	2.544	1.672	-	769.062	877.998	655.067
$\boldsymbol{ASN(\mu,\sigma,\alpha)}$	6.224	1.872	-	1.542	753.012	817.145	724.019
$\boldsymbol{TSN(\mu,\sigma,\lambda)}$	4.83	1.51	2.542	-	711.70	798.06	640.06

Remark: The observed variance-covariance matrix of the MLEs of the parameters

$\widehat{\Theta} = (\hat{\mu},\ \hat{\sigma},\ \hat{\lambda})$ of $TSN(\mu,\ \sigma,\ \lambda)$,distribution from data set are

$$I_n(\widehat{\Theta}) = \begin{bmatrix} 0.1363 & 0.0337 & 0.0347 \\ 0.0328 & 0.0199 & 0.00194 \\ 0.0338 & 0.00194 & 0.0176 \end{bmatrix}$$

Table 2, provides the values of log-likelihood function (Log), Kolmogorov-Smirnov (K-S), P-value. It is evident

Table 2. The statistics -LOG, K-S, P-value for the real data set.

Model	Statistics		
	Log L	K-S	P-value
$N(\mu, \sigma^2)$	-404.919	0.094	0.606
$SN(\mu, \sigma, \lambda)$	-396.161	0.084	0.754
$SL(\mu, \sigma, \lambda)$	-370.531	0.097	0.771
$ASN(\mu, \sigma, \alpha)$	-399.452	0.095	0.740
$TSN(\mu, \sigma, \lambda)$	-385.369	0.073	0.652

From Table 1, Table 2 that, the $TSN(\mu, \sigma, \lambda)$, distribution has the lowest statistics among all fitted models. Hence, this distribution can be chosen as the best model for fitting this data set.

Plots of the estimated density function of the real data set in Figure 4(a):

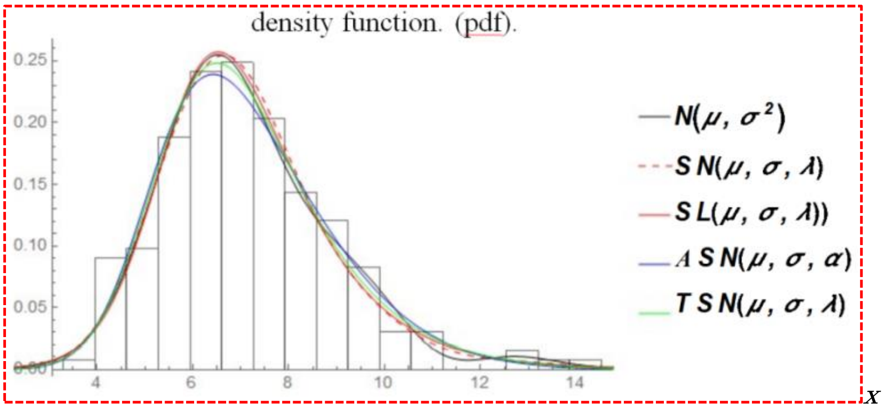

Figure 4(a): the plots of observed and expected densities of some distributions for white cells count (WCC) of 202 Australian athletes.

plots of empirical distribution and estimated cdf for the real data set.

F(x) Cumulative distributions (cdf).

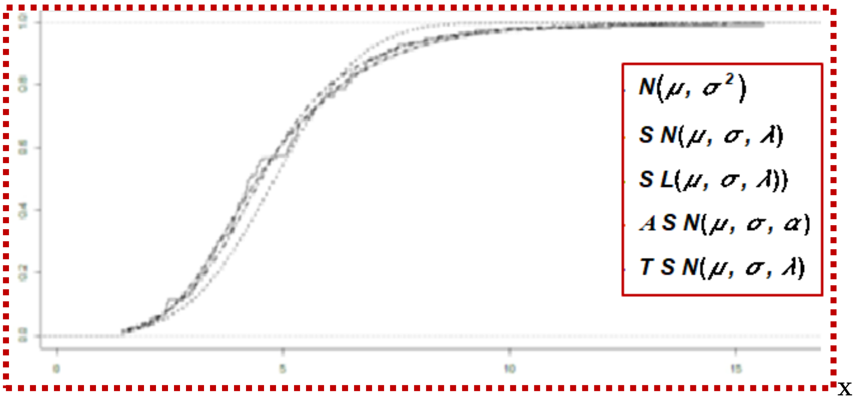

Figure 4(b): the plots of empirical distribution and estimated cdf white cells count (WCC) of 202 Australian athletes.

III. CONCLUSION

A new skew TSN(μ,σ,λ),distribution is constructed using a skew function (which is not a cdf) and some of its distributional properties are studied. The distribution is fitted to real life data sets and is found to perform better in real life data modeling in comparison to the skew-normal SN(μ,σ,λ) distribution Azzalini (1985),Skew logistic SL(μ,σ,λ), Alpha Skew-Normal Distribution $ASN(\mu,\sigma,\alpha)$ and normal N(μ,σ^2)equally well in the other case. Investigation of more skew TSN (μ,σ,λ), distribution considering different skew functions is currently under consideration.

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