© 2019 IJSRST | Volume 6 | Issue 4 | Print ISSN: 2395-6011 | Online ISSN: 2395-602X
Themed Section: Science and Technology
DOI : https://doi.org/10.32628/IJSRST

Similarity Solution of One-Dimensional Flow in Unsaturated Porous Media

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ABSTRACT

In the present paper we have discuss similarity solution for the partial differential equation governing one dimensional unsteady flow through unsaturated porous media. The general starching transformations are employed to derive similarity transformation to transform partial differential equation along with boundary condition into ordinary differential equation. The reduced second order ordinary differential equation is solved by two parameter singular perturbation method.

Keywords : Similarity Analysis, Non-linear Singular perturbation Technique, Porous media.

I. INTRODUCTION

A very large fraction of the water falling as rain on the land surfaces of the earth moves through unsaturated soil during the subsequent processes of infiltration, drainage, evaporation and absorption of soil water by plant roots. Hydrologists have tended, nevertheless, to pay relatively little attention to the phenomenon of water movement in unsaturated soils. Most research on this topic has been done by soil physists concerned ultimately with agronomic or ecological aspects of hydrology.

The present chapter analytically discusses a none dimensional unsteady flow through unsaturated porous media by employing a two parameter singular perturbation technique[1]. The mathematical formulation yields a nonlinear diffusion type equation which transformed into an ordinary differential equation contain two small parameters by employing a similarity method[2].

The mathematical model conforms to the hydrological situation of one dimensional vertical ground water by Spreading [3]. Such flows are of great importance in water resources science, soil engineering and agricultural sciences.

Many researchers have discussed this phenomenon from different aspects, for example Klute[4] and Hank Bower[5] employs a finite difference method; Philips[6] uses a transformation of variable technique; Mehta[7] discussed multiple scale method; Verma[1,8] has obtained Laplace transformation and similarity solution and Sharma [9] discusses a variational approach. The experimental investigations has been discussed by Bruce and Klute [10]; Gardner and Mayhuge[11]; Nielson and Bigger[12]; Rawlins and Gardner[13], Terwilliger [14]; Van Vorts[15] and Rahme [16] have described the phenomena of gravity drainage of liquids through porous media and supported their theoretical investigation by experimental results.

II. STATEMENT OF THE PROBLEM

For definiteness of physical problem, we consider here that the recharge takes place over a large basin of such geological configuration that the sides are limited by rigid boundaries while the bottom is confined by a thick layer of water table. In these circumstances, water will flow vertically downward through unsaturated porous medium. It is assumed that the diffusivity coefficient is equivalent to an average value, over the whole range of moisture content, is small enough to be regarded as a perturbation parameter. Further, the permeability of the medium is considered to vary directly with moisture content and inversely as the square root of times. The mathematical formulation of the basic hydro dynamical equation yields a nonlinear partial differential equation transformation which is derived by using one-parameter stretching transformation. The resulting ordinary differential equation, contains two small parameters, is solved by applying two parameters singular perturbation method.

III. MATHEMATICAL MODEL

The equation of continuity for an unsaturated porous medium is given by

$$\frac{\partial}{\partial t}(\rho_r\theta) = \nabla.\vec{M} \quad (3.1)$$

Where, ρr =the bulk density of the medium

θ = the moisture content on a dry weight basis

$\vec{M}$ = the mass flus of moisture

∇ = the vector differential

t = time in second

It has long been assumed [17] and was confirmed experimentally by Childs and Collis-George[18] and other that Darcy's law may hold for the flow of liquid, water, in an unsaturated media in a modified form in which K is a function of the volumetric moisture content. The theoretical validity of this concept depends on the assumption that the drag at the air-water interfaces in the soil in negligibly small[19]. Thus from Darcy's law for the flow of water in an unsaturated porous medium, including soil, in the modified form we get

$$V = -L(\theta)\nabla\emptyset \quad (3.2)$$

Where ϕ represents the gradient of the whole moisture potential. V is the volume flux of moisture and K, the coefficient of aqueous conductivity combining equations (3.1) and (3.2) we get,

$$\frac{\partial}{\partial t}(\rho_r\theta) = \nabla.(\rho K\nabla\emptyset) \quad (3.3)$$

Where ρ is the fluid density.

Now for an unsaturated porous media, the total potential ϕ may be regarded as comprising moisture potential, Ψ which has been known variously as capillary potential; capillary pressure; moisture tension; moisture suction; negative pressure head; etc., [20-24], and the gravitational component. Z is at the height above some datum level. Thus,

$$\emptyset = \Psi(\theta) + Z \quad (3.4)$$

Substituting (3.4) in the equation (3.3), we then have

$$\rho_r\frac{\partial\theta}{\partial t} = \nabla(\rho K\nabla\Psi) + \frac{\partial}{\partial z}(\rho Kg) \quad (3.5)$$

Recalling that the flow takes place only in the vertical direction, we may write equation (3.5) as

$$\rho_r\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(\rho K\frac{\partial\Psi}{\partial z}\right) - \frac{\partial}{\partial z}(\rho Kg) \quad (3.6)$$

The positive direction of z-axis is the same as that of gravity. Considering θ and Ψ to be connected by a single valued function of θ, we may introduce the quantity D such that

$$D = \frac{\rho}{\rho_r}K\frac{\partial\Psi}{\partial\theta} \quad (3.7)$$

Which is called diffusivity coefficient and therefore we may write (3.6) as

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right) + \frac{\rho_g}{\rho_g}\frac{\partial K}{\partial z} \quad (3.8)$$

Now replacing D by its average value D_a and assuming, as in [7],

$$K = \frac{K_0\theta}{\sqrt{t}} \quad (3.9)$$

Equation (3.8) may be written as

$$\frac{\partial\theta}{\partial t} = D_a\frac{\partial^2\theta}{\partial z^2} + \frac{\rho g k_0}{\rho_r\sqrt{t}}\frac{\partial\theta}{\partial z} \quad (3.10)$$

Considering water table, to be situated at a depth 'L' and setting

$$X=z/L;\ \ T=t/L^2;\ M=\frac{\rho g k_0}{\rho_r} \qquad (3.11)$$

We may write (3.10) as

$$\frac{\partial\theta}{\partial t}=D_a\frac{\partial^2\theta}{\partial z^2}+\frac{M}{\sqrt{T}}\frac{\partial\theta}{\partial x} \qquad (3.12)$$

A set of suitable boundary condition consistent with the physical problem are

$\theta(0,T)=\theta_0$

$\theta(0,T)=\theta$ (3.13)

IV. SIMILARITY ANALYSIS

In this section we analyze the problem of previous section by studding its invariance under global transformations, in particular, stretching transformation, let

$\theta=H\ (X,T)$ (4.1)

By the solution of the problem (3.11-12) we consider the general stretching transformation of (θ,X,T) space as

$\theta^* = \delta\theta$

$X^* = \alpha X$ (4.2)

$T^* = \beta T$

With the parameters (δ, α, β)

If $\delta(\beta)$ and $\alpha(\beta)$ are somehow determined then (4.2) will be one parameter group of transformation. Corresponding to (4.1) we have new surface defined by

$$\theta^* = H^*\ (X^*, T^*) \qquad (4.3)$$

Therefore original solution surface transform

$$\frac{\partial\theta}{\partial T}=\frac{\beta}{\delta}\frac{\partial\theta^*}{\partial T^*} \qquad (4.4)$$

$$\frac{\partial\theta}{\partial X}=\frac{\beta}{\delta}\frac{\partial\theta^*}{\partial X^*} \qquad (4.5)$$

$$\frac{\partial\theta}{\partial T}=\frac{\beta}{\delta}\frac{\partial\theta^*}{\partial T^*} \qquad (4.6)$$

Since $\theta(X,T)$ is defined by (3.12), we have,

$$\frac{\beta}{\delta}\frac{\partial\theta^*}{\partial T^*}=D_a\frac{\alpha^2}{\delta}\frac{\partial^2\theta^2}{\partial X^{*2}}+\frac{M\sqrt{\beta}}{\sqrt{T^*}}\frac{\alpha}{\delta}\frac{\partial\theta^*}{\partial X^*} \qquad (4.7)$$

For invariance it is necessary that both the operator on the left and the right hand side of (4.7) agree with those in (3.11) multiplied by a common factor. Thus for invariance,

$\delta=1,\ \alpha^2=\beta\Rightarrow\alpha=\sqrt{\beta}$

Therefore (4.2) becomes

$\theta^* = \theta$

$X^* =\sqrt{\beta}X$ (4.8)

$T^* = \beta T$

Thus (4.8) is the group transformations leaving problem (3.11) invariant under this transformation. The boundary condition attached to (3.11) is also invariant. Thus for θ^* we have

$$\frac{\partial\theta^*}{\partial T^*}=D_a\frac{\partial^2\theta}{\partial X^{*2}}+\frac{M\sqrt{\beta}}{\sqrt{T^*}}\frac{\partial\theta^*}{\partial X^*} \qquad (4.9)$$

$\theta^*(0,T^*) = \theta_0$

$\theta^*(1,T^*) = 1$ (4.10)

Now due to uniqueness, must be same function of (X^*, T^*) as θ^* is of (X,T).

This is

$$\theta^*(X, T) = \theta(X^*, T^*) \qquad (4.11)$$

As a consequence of transformation (4.8) and the invariance condition (4.11) we, thus, obtained a functional equation which must be satisfied by solution. Therefore (4.11) implies.

$$\theta(\sqrt{B}X,\beta T)=\theta(X,T) \qquad (4.12)$$

This functional relation (4.12) holds true for all values of β. To obtain a functional form that the solution $\theta(X, T)$ must have consider of $\frac{\partial}{\partial\beta}$ (4.12) near the identity $\beta=1$. We therefore, have

$$\frac{X}{2\sqrt{\beta}}\frac{\partial}{\partial X}\theta(\sqrt{\beta}X,\beta T)+T\frac{\partial}{\partial T}\theta(\sqrt{\beta}X,\beta T) \qquad (4.13)$$

As $\beta\rightarrow$we see that $\theta(X,T)$ must satisfies a first order partial differential equation

$$\frac{X}{2}\frac{\partial}{\partial X}\theta(X,T)+T\frac{\partial}{\partial T}\theta(X,T)=0 \qquad (4.14)$$

This enables the form of the solution to be found. Since the general solution of (4.14) involves an arbitrary function, the characteristic equations associated with (4.14) are

$$\frac{dX}{X/2} = \frac{dT}{T} = \frac{d\theta}{0} \quad (4.15)$$

The integral of the first two in (4.15) is

$$\eta = \frac{X}{\sqrt{T}} \quad (4.16)$$

And that of for the last two of (4.15) is

θ = constant = F(η).

Thus the general solution of (4.15) has the form

$$\theta(X,T) = F(\eta); \ \ \eta = \frac{X}{\sqrt{T}} \quad (4.17)$$

V. SOLUTION OF THE PROBLEM

In this section we discuss the existence of weak solutions (25) for unidimensional flow in unsaturated media, from rigorous mathematical view point which is the recent research interest at the international level [26-29]. Applying the result (3.9) and (3.11) to equation (3.8), we get

$$\frac{\partial\theta}{\partial T} = \frac{\partial}{\partial X}\left|D(\theta)\frac{\partial\theta}{\partial X}\right| + \frac{M}{\sqrt{T}}\frac{\partial\theta}{\partial X} = 0 \quad (5.1)$$

Now applying the similarity functional relation (4.17), equation (5.1) will become a nonlinear ordinary differential equation.

$$\frac{d}{d\eta}\left[D(F)\frac{dF}{d\eta}\right] + \left(M + \frac{1}{2}\eta\right)\frac{dF}{d\eta} = 0 \qquad 0 < \eta < \infty \quad (5.2)$$

We impose the conditions at the boundaries

$$F(0) = \theta_0\ ; \qquad \lim_{n\to\infty} F^n = 0 \quad (5.3)$$

In the physical variable this means that for fixed T,

$$\theta(0,T) = \theta_0\ ; \qquad \lim_{X\to\infty} \theta(X,T) = 0 \quad (5.4)$$

Equation (5.4) implies initial saturation is constant but as $X\to\infty$, saturation of moisture will reduces and finally it vanishes.

For the existence of a positive solution (5.2) and (5.3) we have by a uniqueness arugment F<0 [Cf.30, lemma2].

Hence D(F) F'<0 and, in view of (5.2), increasing. Since D(F)F' cannot tend to a negative limit as $\eta\to\infty$, it follows that D(F)F'=0. Integrating equation (5.2) from η to ∞, we get

$$D[F(\eta)]F'(\eta) = \int_{\eta}^{\infty}\left(M + \frac{1}{2}\Psi\right)F'(\Psi)d\Psi \quad (5.5)$$

In view of strict monotonicity of F, we define an inverse transformation,

$$\eta=\sigma(F)\Rightarrow \quad (5.6)$$

$$1 = \frac{d\sigma}{dF}\frac{dF}{d\eta} \quad (5.7)$$

Therefore equation (5.7) $\Rightarrow$

$$\frac{d\sigma}{dF} = \frac{1}{dF/d\eta} \quad (5.7a)$$

Using the result (5.6), (5.7-7a) in (5.5), we get

$$\frac{d\sigma}{dF} = -D(F)/\int_0^F\left(\frac{1}{2}\sigma + M\right)d\emptyset \quad (5.8)$$

Equation (5.8) is an integro differential equation which attached conditions

$$\sigma(\theta_0) = 0\ \ ; \quad \lim_{F\to 0}\sigma(F) = \infty \quad (5.9)$$

Setting

$$g(F) = \int_0^F\left(\frac{1}{2}\sigma(\emptyset) + M\right)d\emptyset \quad (5.10)$$

Therefore equation (5.8) becomes

$$gg'' = -\frac{1}{2}D(F) \quad (5.11)$$

Since g(0) = 0, g'(θ_0), $\operatorname{limit}_{F\to 0} g'(F) = \infty$

Now suppose $\operatorname{limit}_{F\to 0} Sup.\, g'^{(F)} = A < \infty$

then from equation (5.10), we have,

g"(F)<0 on (0, θ_0).

Hence g<λF on (0, θ_0).

Thus

$$\lim_{F\to 0}\int_F^{\theta_0}\frac{D(S)}{S}dS < 2A^2 < \infty \quad (5.12)$$

This verifies the condition for the existence of weak solution equating (5.1) and (5.2). We define

$$u(F)=[g'(F)-M]^2 \quad (5.13)$$

The existence of solution of an integro differential equation (5.8) to gather with the condition (5.9) implies the existence of solution of original problem (5.1).

Differentiating equation (5.13) and eliminating g"(F) with the help of (5.11) and integrating the resulting equation, we get

$$u(F) = -\int_F^{\theta_0}\left[\frac{D(F)}{t+\frac{1}{2}\int_0^t\frac{D(S)}{S}(u+M)^2 dS} + \frac{BD(t)}{\int_0^t(\sqrt{u}+M)dS}\right]dt \quad (5.14)$$

The uniqueness is intended to be discussed subsequent by using shooting technique [28].

VI. CONCLUSION

Similarity solution for the partial differential equation governing one dimensional unsteady flow through unsaturated porous media is derived using general group of transformation yield second order ordinary differential equation of boundary value type. Analytical solution of this equation can be obtained by two parameter singular perturbation method.

VII. REFERENCES

[1]. O'Malley, R.E., (1966): Two parameter singular perturbation problems, Ph.D. Dissertation, Standrord, University, Univ. Microfilms, Inc. Ann. Arbor.

[2]. Blumann, G. Wand Cole, J.D., (1974): Similarity methods for differential equation. WSpringer-Verlag, Berlin.

[3]. Verma A.P., (1969): The Laplace transform solution of a one dimensional groundwater recharge by spreading, Annali Di Geofisical, 22,1, p.25.

[4]. Klute, A., (1952): A numerical method for solving the flow equation of water in unsaturated materials, Soil sciences, 73,2,p.105.

[5]. Hanks R.J., and Bowers, S.A(1962): Numerical solution for the moisture flow equation for infiltration into layered soil, Soil Sci. Soc. Am.Proc. 26,pp530-35.

[6]. Phillips, R., (1970): Advances in hydrosciences, Ed. Ven Fe Chow, Acad P. Vol6 New York.

[7]. Mehta, M.N.,(1976): A multiple scale solution of 1-D flow in unsaturated porous media, Indiaqn J. Engg.Maths.

[8]. Verma, A.O., and Mishra, S.K., (1973): A similarity solution of unidimensional vertical ground water recharge through porous meida, Rev. Roum. Scien. Techn. Ser. Mec. Appl18,2,p.345.

[9]. Sharma, S.V.K., (1965): Problem of Partially saturated unsteady state of flow through porous media with special reference to groundwater recharge by spreading., J.Sci. Engg. Research, 9, 69.

[10]. Bruce, R.R., and Klute, A., (1956): The measurement of soil moisture Diffusivity, Soil Sci. Soc. Amer. Proc. 20,pp458-62.

[11]. Gardner, W.H., and Mayhug M.S., (1958): Solutions and tests on the diffusion equation for the movement of water in soil, Soil Sci. Amer. Proc.22.

[12]. Nielson, D.R., et al., (1962): Experimental consideration of diffusion analysis in unsaturated flwo problems, Soil Sci. Amer Proc.26,ppR107-112

[13]. Rawlins, S.L., and Gardner W.H., (1963): The test of the validity of the diffusion equation for unsaturated flow of soil water, Soil Sci. Amer Proc.25pp507-10

[14]. Terwilliger, R.D., (1952: An Experimental and theoretical investigation of gravity drainage performance, trans AIME,p.195

[15]. Van Vorts, W.D., (1933): The international mechanism of the draying of Granual material, Ph.D. Thesis, V.C. lea Angelos.

[16]. Rahme, H.S. (1954): Gravity drainage of liquids from porous media, Ph.D. thesis, V.C. les Angelos.

[17]. Richards, L.A., (1931): Capillary conduction of liquids through porous mmedia, Physics-1, pp.318-33

[18]. Chidls, E.C., and Collis-George, N., (1950: The permeability of porous materials, Proc. Roy. Soc.A201, pp.392-405

[19]. Philip, J.R., (1957): Remarks on the analytical derivation of the Darcy's equation. Trans.Am.Geophys. Union, 38,pp 728-84.

[20]. Rechards, L.A., (1949): Method of measuring soil moisture tension, Soil Sci. 68,pp.95-112.

[21]. Richards, L.A., (1953): Water conducting and retaing properties of soil in relation to irrigation, Desert Research Res. Council, Israel, Spec.Publ 2, pp.523-46.
[22]. Childs, E.C., and Collis-George, N., (1950):The control of soil water, Agron. 2, pp 233-72.
[23]. Croney, D.et al., (1952): The solution of moisture held in soil and other porous material, Road, Res. Tech. Paper 24.
[24]. Miller, E.E., and Klute, A., (1967): The dynamics of soil water part-I, Mechanical forces in irrigation of Agricultural land, Am. Soc. Agron. Madison, Wisconsin,pp209-44.
[25]. Jeffrey, A., and Taniuti, T. (1904): Nonlienar wave Propogation with application to physics and managetohydro dynamics, Academic Press, New York p. 166.
[26]. Baiocchi, C., (1974): Free boundary problems in the theory of fluid fliw through porous media , Proc. Int. Cong. of Mathematician, Vancour, Canada, Aug.21-29.
[27]. Baiocchi, C., (1973): Free boundary problems in the theory of fluid fliw through porous media – Existence and uniqueness theirems, Annali di Mathematica Pure and Applicata, Vol. XCVII pp.1-82.
[28]. Peletier, L.A., and Atkinson, F.V., (1974): Similarity solutions of Non-Linear diffusion equation, Arch. Rat. Mech. Anal.54, 373.
[29]. Aronson, D.G., (1971): Asymtotic behavior of porous media equation, Arch. Rat. Mech. Anal.42.

Cite this article as :

Dharmendra G. Patel, Dr. Ritu Khanna, Dr. P.H. Bhathawala, "Similarity Solution of One-Dimensional Flow in Unsaturated Porous Media", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 6 Issue 4, pp. 401-406, July-August 2019.
Journal URL : http://ijsrst.com/IJSRST19673