Similarity Analysis of Instability Phenomenon in Porous Media with Mean Capillary Pressure

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ABSTRACT

In the present paper, we analytically discussed the phenomenon of instability in polyphasic flow through porous media with the assumption of mean pressure. We consider that water is injected at a uniform rate into an oil-saturated porous medium, and at the interface between the two fluids, there arise protuberance of water into oil zone at relatively greater speeds, compared to the average movement of the interface. In this paper, we have applied a continuous group transformation technique of a similarity analysis, which reduces the governing non-linear partial differential equation to a non-linear ordinary differential equation. The existence and uniqueness of solution of this reduced solution is indicated. A perturbation solution of equation is obtained using the method of composite expansions.

Keywords : Non-Linear Ordinary Differential Equation, Homogeneous Medium, Linear Functional

I. INTRODUCTION

The phenomenon of instability in polyphasic flow through homogeneous porous media without capillary pressure was examined from a statistical view point by Sheidegger and Johnson [1]. Verma [2] has explained the behavior if instability in a displacement precess through heterogeneous porous medium. The present paper analytically discussed the phenomenon of instability in polyphasic flow through a homogeneous medium with mean capillary pressure [3]. The non-linear partial differential equation governing the phenomenon has been reduced to a non-linear ordinary differential equation by the continuous group transformation technique of similarity analysis [4]. The existence and uniqueness of the solution of the problem has also been indicated [5]. A perturbation solution has been obtained, which should give consistent approximation under the conditions of uniqueness.

II. STATEMENT OF THE PROBLEM

We consider here that there is a uniform water injection into an oil saturated porous medium of homogeneous physical characteristics, such that he injected water cuts through the oil formation and gives rise to protuberances (instability), thus furnishing a well developed fingers flow [6]. Here a unidimensional flow is considered, x indicating the co-ordinate in the direction of flow with the origin at the interface. Due to the pressure of a large formation of water at he boundary x=0, it is assumed that the water saturation at this boundary is almost equal to 1, and this initial saturation is further assumed to remain constant during this displacement process. Our particular interest in this paper is to explore the

possibilities of transforming the system of partial differential equation along with initial and boundary conditions, that govern this phenomenon, into an ordinary differential equation with suitable conditions, by the application of similarity analysis and to indicate the existence and uniqueness of the reduced system, which is reasonably known. Under the conditions for which the uniqueness of solution, thus insuring the reasonableness of the approximation.

III. FUNDAMENTAL EQUATION

The seepage velocity of water (V_w) and oil (V_o) are given by Darcy's law as,

$$V_W = -\left(\frac{k_w}{\delta_w}\right) \cdot K \left[\frac{\partial p_W}{\partial x}\right] \quad (3.1)$$

$$V_o = -\left(\frac{K_o}{\delta_o}\right) \cdot K \left[\frac{\partial P_o}{\partial x}\right] \quad (3.2)$$

Where K is the permeability of the homogeneous medium, k_w and k_o are the relative permeabilities of water and oil, which are functions of respective saturations of water and oil S_w and S_o. p_w and p_o are pressures in the water and oil phases. δ_w and δ_o are kinetic velocities.

The equation of continuity of the two phases are

$$P \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \quad (3.3)$$

$$P \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \quad (3.4)$$

Where P is the porosity of the medium. From the definition of phase saturation, it is evident that,

$$S_w + S_o = 1 \quad (3.5)$$

CAPILLARY PRESSURE:

The capillary pressure $'p_C'$ is defined as the pressure discontinuity between the flowing phases across their common interface and is a function of the phase saturation. We assume a continuous linear functional relation of the form,

$$p_c = -\beta S_w \quad (4.1)$$

$$p_c = p_0 - p_w \quad (4.2)$$

Where β is constant.

RELATIVE PERMEABILITIES:

For definiteness of the mathematical analysis, we assume standard relationships, due to Scheidegger and Johnson [1] between phase saturations and relative permeabilities as

$$k_w = S_w, \quad k_o = 1 - S_w = S_o \quad (5.1)$$

FORMULATION OF DIFFERENTIAL SYSTEM:

To derive the equation of motion for saturation, we substitute the value of V_w and V_o from equations (3.1-2) into equations (3.3-4) respectively, getting

$$P \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_w}{\delta_w} K \frac{\partial P_W}{\partial x}\right] = 0 \quad (6.1)$$

$$P \frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_o}{\delta_o} K \frac{\partial P_p}{\partial x}\right] = 0 \quad (6.2)$$

Eliminating $\frac{\partial P_w}{\partial x}$ from equations (6.1) and (4.2), we get,

$$P \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left[K \frac{k_w}{\delta_w}\left(\frac{\partial p_o}{\partial x} - \frac{\partial P_c}{\partial x}\right)\right] \quad (6.3)$$

Combining equations (6.2) and (6.3) and using equation (3.5), we get

$$\frac{\partial}{\partial x}\left[K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)\frac{\partial p_o}{\partial x} - \frac{k_w}{\delta_w}K\frac{\partial P_c}{\partial x}\right] = 0 \quad (6.4)$$

Integrating (6.4) with respect to x, we get,

$$K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)\frac{\partial p_o}{\partial x} - \frac{k_w}{\delta_w}K\frac{\partial P_c}{\partial x} = -V \quad (6.5)$$

Where V is constant of integration, whose value will be determined by later analysis.

Equation (6.5) can be rearranged as,

$$\frac{\partial p_o}{\partial x} = -\frac{V}{K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} + \frac{\left(\frac{\partial P_c}{\partial x}\right)}{1 + \frac{k_o \cdot \delta_w}{k_w \cdot \delta_o}} \quad (6.6)$$

Using (6.6) in (6.3), we obtain,

$$P\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\frac{k_o}{\delta_o}\frac{\partial p_c}{\partial x}}{1+\frac{k_o}{k_w}\cdot\frac{\delta_w}{\delta_o}} + \frac{V}{1+\frac{k_o}{k_w}\cdot\frac{\delta_w}{\delta_o}}\right] = 0 \qquad (6.7)$$

The value of the pressure of oil(p_o) can be written, as in [7], in the form

$$p_o = \frac{1}{2}(p_o + p_w) + \frac{1}{2}(p_o - p_w) = \bar{p} + \frac{1}{2}p_c \quad (6.8)$$

where $\bar{p}$ is the mean pressure which is constant.

Using (6.8) in (6.5), we get

$$V = \frac{K}{2}\left[\frac{k_w}{\delta_w} - \frac{k_o}{\delta_o}\right]\frac{\partial P_c}{\partial x} \qquad (6.9)$$

Substituting this value of V in (6.7), we get,

$$P\frac{\partial S_w}{\partial T} + \frac{1}{2}\frac{\partial}{\partial x}\left[K\left(\frac{k_w}{\delta_w}\right)\frac{\partial p_c}{\partial S_w}\frac{\partial S_w}{\partial x}\right] = 0 \qquad (6.10)$$

Substituting the linear functional form of p_c and k_w from (4.1) and (5.1), and setting $X = \frac{x}{L}, T = \frac{t}{L^2}$, where L is a constant reference length, equation (6.10) reduced to

$$P\frac{\partial S_w}{\partial T} - \frac{\beta K}{2\delta_w}\frac{\partial}{\partial x}\left[S_w\frac{\partial S_w}{\partial x}\right] = 0 \qquad (6.11)$$

Equation (6.11) is the desired non-linear partial differential equation of motion for water saturation that governs the flow of two immiscible phases in a homogeneous porous medium. We can write down the relevant initial and boundary conditions associated with the description of the above model as,

$$S_w(X, 0) = 0, \ S_w(0, T) = S_{w_0} < 1, \ \lim_{X\to\infty} S_w(X, T) = 0 \qquad (6.12)$$

SIMILARITY ANALYSIS:

The similarity analysis of the equation (6.11) is done by one-parameter continuous group transformation method [4]. (6.11) can be put in the form

$$P\frac{\partial S_w}{\partial T} + Q\left(\frac{\partial S_w}{\partial X}\right)^2 + Q\, S_w \frac{\partial^2 S_w}{\partial X^2} = 0 \qquad (7.1)$$

Where $Q = -\left(\frac{K\beta}{2\delta_w}\right)$

Let $S = \theta(X, T)$

by the solution of the problem (7.1). consider the continuous group of transformations of (S, X, T) – space in the form,

$$S_w^* = vS, \qquad X^* = \alpha X, \qquad T^* = \beta T \qquad (7.3)$$

with parameter (v, α, β). If $v(\beta)$ and $\alpha(\beta)$ are somehow determined, then (7.3) will be a family of one parameter (β) continuous group of transformations and a new solution surface corresponding to (7.2) is.

$$S_w^* = \theta^*(X^*, T^*) \qquad (7.4)$$

$$\left.\begin{array}{l} \frac{\partial S_w}{\partial T} = \frac{\beta}{v}\frac{\partial S_w^*}{\partial T^*} \\[2mm] \frac{\partial S_w}{\partial X} = \frac{\alpha}{v}\frac{\partial S_w^*}{\partial X^*} \\[2mm] \frac{\partial^2 S_w}{\partial X^2} = \frac{\alpha^2}{v^2}\frac{\partial^2 S_w^*}{\partial X^{*2}} \end{array}\right\} \qquad (7.5)$$

Substituting these values in (7.1), we get

$$\frac{\beta}{v}\left(P\frac{\partial S_w^*}{\partial T^*}\right) + Q\frac{\alpha^2}{v^2}\left(\frac{\partial S^*}{\partial X^*}\right)^2 + Q\frac{S_w^*}{v}\frac{\alpha^2}{v}\frac{\partial^2 S^*}{\partial X^{*2}} = 0 \ (7.6)$$

Thus, for invariance of (7.1) under the transformations, we must have

$$v = 1; \quad \alpha^2 = \beta \Rightarrow \alpha = \sqrt{\beta}.$$

Hence, (7.3) becomes,

$$S_w^* = S_w, \ X^* = \sqrt{\beta}X, \ T^* = \beta T \qquad (7.7)$$

Thus, (7.7) is the required continuous group of transformations which leaves the equation (7.1) invariant. Hence, for s_w^*, we have,

$$P\frac{\partial S_w^*}{\partial T^*} + Q\left(\frac{\partial S_w^*}{\partial X^*}\right)^2 + QS_w^*\frac{\partial^2 S_w^*}{\partial X^{*2}} = 0 \qquad (7.8)$$

Now, due to uniqueness, θ must be same function of (X^*, T^*) as θ^* is of (X, T), i.e.,

$$\theta^*(X,T) = \theta(X^*,T^*) \qquad (7.9)$$

As a consequence of transformation (7.7) and the invariance condition (7.9), we obtain a functional equation which must be satisfied by the solution; therefore, (7.9) implies,

$$\theta\left(\sqrt{\beta}X, \beta T\right) = \theta(X,T) \qquad (7.10)$$

Now, $\partial/\partial\beta$ of (7.10) implies,

$$\frac{X}{2\sqrt{\beta}}\frac{\partial}{\partial X}\left[\theta\left(\sqrt{\beta}\,X, \beta T\right)\right] + T\frac{\partial}{\partial T}\left[\theta\left(\sqrt{\beta}X, \beta T\right)\right] = 0$$

as $\beta \to 1, \theta(X,T)$ satisfies a first order partial differential equation

$$\frac{X}{2}\frac{\partial}{\partial X}\theta(X,T) + T\frac{\partial}{\partial T}\theta(X,T) = 0 \qquad (7.12)$$

The general solution of (7.12) involves an arbitrary function. The characteristic equation associated with (7.12) are

$$\frac{dX}{\left(\frac{X}{2}\right)} = \frac{dT}{T} = \frac{d\theta}{O} \qquad (7.13)$$

The integral of the first two in (7.13) implies $\eta = \frac{X}{\sqrt{T}}$

and that of for the last pair gives

$$\theta(X,T) = constant = F(\eta) \quad \text{i.e. } S_w = F(\eta) (7.15)$$

Therefore, the general solution to (7.13) is of the form

$$S_w = F(\eta); \quad \eta = \frac{X}{\sqrt{T}} \qquad (7.16)$$

Functional relation (7.16) gives the similarity transformation which is determined by using the one-parameter continuous group o transformations (7.7). This similarity transformation reduces the equation (7.1) to an ordinary differential equation

$$(FF')' - \frac{P}{2Q}\frac{1}{2}\,\eta\,F'(\eta) = 0 \qquad (7.17)$$

Where the primes denote differentiation with respect to η

Substituting the value of Q here, we get,

$$[F(\eta)F'(\eta)]' + \frac{P\,\delta_w}{K\beta}\,\eta\,F'(\eta) = 0 \qquad (7.18)$$

This similarity transformation is compatible with our original initial boundary value problem, as we see that the set of conditions (6.12) get transformed to

$$F(0) \overset{(7.11)}{=} S_{w\,0} < 1. \quad \text{Lim}_{\eta\to\infty}\ F(\eta) = 0 \qquad (7.19)$$

UNIQUENESS AND EXISTENCE OF SOLUTIONS:

In this section, we mention the results on the uniqueness and existence of solutions of the problem (7.18-19). We rewrite the system as,

$$\left.\begin{array}{l} \frac{d}{d\eta}\left[F(\eta)\frac{dF}{d\eta}\right] + \ln\frac{dF}{d\eta} = 0 \\ F(0) = S_{w_o}, \lim_{\eta\to\infty} F(\eta) = 0 \end{array}\right\} \qquad (8.1)$$

The system (8.1) is a special case of more general problem discussed in [8], where one of the authors has discussed (7.14) aspects of similar problems.

The uniqueness and existence of a weak solution with compact support for the problem

$$\left.\begin{array}{c} \frac{d}{d\eta}\left[F^m\frac{dF}{d\eta}\right] + \ln\frac{dF}{d\eta} = qF, \quad 0 < \eta < \infty \\ F(0) = F_0, \quad F(\infty) = 0 \end{array}\right\} \qquad (8.2)$$

has been discussed by Gilding and Peletier [9]. There is was proved that if $F_0 > 0, 1 \geq O$, and $2l + q > 0$, then there exists a unique $a > 0$ and a unique solution of the problem (8.2) such that $F(\eta) > 0$ in $0 < \eta < a$ and $F(\eta) = 0,\ a < \eta < \infty$. In equation (8.1), we have $l = \left(\frac{P\delta_w}{K\beta}\right) > 0, q = 0$ and $F_o > 0$. Hence, the prerequisites for applying the results of the above mentioned reference are satisfied. Also, we know that all classical solutions are weak solutions. Hence, the uniqueness of a classical solution, if it exists, is assured. The existence of

classical solution is under investigation by the authors.

PERTURBATION SOLUTION

In this section we find an approximate solution of the problem (7.17), using the singular perturbation technique of composite expansions. Substituting the value of Q in equation (7.17) it becomes,

$$F F'' + F'^2 + \lambda \eta F' = 0 \qquad (9.1)$$

Where $\lambda = \frac{P\delta_w}{K\beta}, F = F(\eta)$, and the prime denotes a differentiations with respect to η.

Corresponding to the transformation (7.15-16), the initial and boundary conditions given by (6.12) become

$$F(0) = S_{w_o}; \ \lim_{\eta \to \infty} F(\eta) = 0 \qquad (9.2)$$

Setting $F(\eta) = 1 - \theta f(\eta); 0 < \delta = 1 - S_{w_o} < 1$, equations (9.1-2) become

$$(1 - \theta f)f'' - \theta f'^2 + \lambda \eta f' = 0 \qquad (9.3)$$

$$f(0) = 1, \ f(\infty) = \alpha(\neq 0) \text{ say} \qquad (9.4)$$

Keeping in mind the physical situation of the problem we take $'\theta'$ as a perturbation parameter. To find the perturbation solution we employ the method of composite expansions [10].

Let

$$f = \sum_{r=0}^{\infty} \theta^r f_r(\eta) \qquad (9.5)$$

be the solution of the problem (9.3). Substituting (9.5) into equation (9.3) and equation the coefficients of each of θ^0 and θ^1 on both sides of the resulting equation, we get

$$f_0'' + \lambda \eta f_0' = 0 \qquad (9.6)$$

$$f_1'' + \lambda \eta f_1' = f_0 f_0'' + f_0'^2 \qquad (9.7)$$

Now, the solution to equation (9.6) is,

$$f_0(\eta) = A \int_0^\eta exp.\left(-\frac{1}{2}\lambda \theta^2\right) d\theta + B \quad (9.8)$$

Using the conditions given by (9.4) in (9.8), it becomes

$$f_0(\eta) = 1 + (\alpha - 1)erf.\left(\eta \sqrt{\frac{\lambda}{2}}\right) \qquad (9.9)$$

Substituting the value of $f_0(\eta)$ from (9.9) in (9.7), it becomes

$$f_1'' + \lambda \eta f_1' = (\alpha - 1)^2 \ \pi^{-1}2\lambda \ (\lambda \eta + 1)exp.(-\lambda \eta^2) \ (9.10)$$

Equation (9.10) is a linear differential equation in $f_1'(\eta)$ and its solution is obtained as

$$f_1'(\eta) = (\alpha - 1)\left(\frac{2\lambda}{\pi}\right)\left[exp.\left(-\frac{1}{2}\lambda \eta^2\right) - exp.(-\lambda \eta^2) + \sqrt{\frac{\pi}{2}} \ erf.\left(\eta \sqrt{\frac{\lambda}{2}}\right) exp.\left(-\frac{1}{2}\lambda \eta^2\right)\right] + c \cdot exp.\left(-\frac{1}{2}\lambda \eta^2\right) \qquad (9.11)$$

Where C is constant of integration. Integrating (9.11) w.r.t. η and simplifying the result with approximation for $exp.\left(-\frac{1}{2}\lambda \eta^2\right)$, we have

$$f_1(\eta) \cong (\alpha - 1)\sqrt{\frac{2\lambda}{\pi}} \left[erf.\left(\eta \sqrt{\frac{\lambda}{2}}\right) - \left(\frac{1}{\sqrt{2}}\right) \cdot erf.\left(\eta \sqrt{\lambda}\right) + \frac{1}{2}\eta^2 - \frac{1}{6}\eta^4\right] - \frac{(\alpha-1)\sqrt{2\lambda}}{\sqrt{\pi}} \cdot \left[\frac{\sqrt{2}-1}{\sqrt{2}} + \frac{\pi}{2}\right] erf.\left(\eta \sqrt{\frac{\lambda}{2}}\right) + 1 \qquad (9.12)$$

Substituting (9.9) and (9.12) into (9.5), we get,

$$f_1(\eta) \cong 1 + (\alpha - 1)erf.\left(\eta \sqrt{\frac{\lambda}{2}}\right) + \theta \ \left[(\alpha - 1)\sqrt{\frac{2\lambda}{\pi}} \left[erf.\left(\eta \sqrt{\frac{\lambda}{2}}\right) - \frac{1}{\sqrt{2}}erf.(\sqrt{\lambda} \cdot \eta) + \frac{1}{2} \eta^2 - \frac{1}{6}\eta^4\right] - (\alpha - 1)\sqrt{\frac{2\lambda}{\pi}} \left(\frac{\sqrt{2}-1}{\sqrt{2}} + \frac{\pi}{2}\right) \cdot erf.\left(\eta \sqrt{\frac{\lambda}{2}}\right) + 1 \ \right] \qquad (9.13)$$

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