International Journal of Scientific Research in Science and Technology
Print ISSN: 2395-6011 | Online ISSN: 2395-602X (www.ijsrst.com)

Existence and Uniqueness Theorem for the Flow of two Immiscible Fluids Through Porous Media with Decreasing Exponential Saturation

Pandya Parth M[1], Dr. Gajendra Purohit[2], Dr. P. H. Bhathawala[3]
[1]Research Scholar, Pacific University, Udaipur, Rajasthan, India
[2]Director, Pacific College of Basic & Applied Sciences, Pacific University, Udaipur, Rajasthan, India
[3]Professor, Department of Mathematics, V.N.S.G. University, Surat, Gujarat, India

ABSTRACT

Article Info
Volume 7, Issue 6
Page Number: 472-482
Publication Issue :
November-December-2020

Article History
Accepted : 15 Dec 2020
Published : 30 Dec 2020

We are able to find a wide variety of physical phenomenon from the domain of natural sciences in which exponential decays occurs. In the present paper, we have considered the decreasing exponential saturation function for the flow of two immiscible fluids through porous media. The partial differential equation arises for the flow of two immiscible fluids through porous medium with decreasing exponential saturation yields a non-linear partial differential equation of parabolic nature. Such equations are very difficult to solve analytically. The present paper describes the existence and uniqueness of similarity of this type of equations.

Keywords : Physical Phenomenon, Linear Partial Differential Equation

I. INTRODUCTION

The non-linear partial differential system governing the flow of two immiscible fluids through porous media, as in [1] is given by,

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial x}\left| R(S)\frac{\partial S}{\partial x}\right| \tag{1.1}$$

and the corresponding boundary and initial conditions are

$$s(x,0) = 0 \tag{1.2}$$

$$s(0,t) = f(t) \tag{1.3}$$

$$\lim_{x\to\infty} s(x,t) = 0 \quad \text{for } t > 0 \tag{1.4}$$

where $s > 0,\ 0 < x < \infty,\ 0 < t \le T$ and

$$R(s) = \frac{K}{P}\cdot\frac{\frac{k_i}{\delta_i}\frac{k_n}{\delta_n}}{\frac{k_i}{\delta_i}+\frac{k_n}{\delta_n}}\cdot\frac{dP_c}{ds}$$

in which

K = Permeability of the media
P = Porosity of the media
K_i = Relative permeability of the injected phase

k_n = Relative permeability of the native phase
δ_i = Viscosity of the injected phase
δ_n = Viscosity of the native phase
s = Saturation of the injected phase
t = time
x = special co-ordinate
P_c = Capillary pressure

Equation (1.1) is parabolic at any point (x, t), at which $s > 0$. However at points where $s = 0$, it is degenerate parabolic. Because of this degeneracy, (1.1) need not always have a classical solution.

A class of weak solution of (1.1) were introduced by Oleinik, Kalashnikov and You-Lin [2]. They proved existence and uniqueness of such solutions and in addition they showed that if at some instant $'t_0'$, a weak solution of $s(x, t_0)$ has a compact support, then $s(x, t)$ has compact support for any $t \geq t_0$.

Equation (1.1), for $R(s) = \lambda e^{-\alpha}$, $f(t) = f_0 e^{-\alpha}$ is transformed into an ordinary differential equation,

$$(f^v f')' + \frac{v\alpha+1}{2\lambda}\eta f' - \frac{\alpha}{\lambda} f = 0 \tag{1.5}$$

with the help of similarity transformation

$$\eta = \frac{x}{t^{\frac{\alpha+1}{2}}}, \qquad s = e^{-\alpha} f(\eta); \quad 0 < \eta < \infty$$

Where λ, v, α are constants and $(v, \alpha) > -1$, and dashes denote differentiation w.r.t. η.

At the boundaries, we require the condition,

$$f(0) = f_0$$
$$f(\infty) = 0 \qquad \text{for fixed } t \in [0, T]$$

The rigorous study of these similarity analysis was done by Atkinson and Peletier [3,4] and by Shampine [5,6]. They considered the equation,

$$[k(f)\, f']' + \frac{1}{2}\eta f' = 0, \qquad 0 < \eta < \infty \tag{1.6}$$

in which $k(s)$ is defined, real and continuous for $s > 0$ with $k(0) \geq 0$ and $k(s) > 0$ if $s > 0$. Clearly, if we set $\alpha = 0$, equation (1.5) becomes a special case of (1.6).

In this paper, we extend the analysis of [3] to problem

$$[f^m]'' + p\eta f' = qf \qquad 0 < \eta < \infty \tag{1.7}$$
$$f(0) = f_0, \quad f(\infty) = 0 \tag{1.8}$$

where $p = \frac{v\alpha+1}{2\lambda}$, $q = \frac{\alpha}{\lambda}$ in which α, λ, v are arbitrary constants.

Obviously equation (1.7) incorporates equation (1.5) and therefore, it is necessary to consider a weak solution of the problem (1.7), (1.8).

DEFINITION

A function f is said to be a weak solution of equation (1.7),(1.8) if,

(i) f is bounded, continuous, and non-negative on $[0, \infty)$.
(ii) $(f^m)(\eta)$ has continuous derivative w.r.t. η on $(0, \infty)$ and
(iii) f satisfies the identity

$$\int_0^\infty \phi'\{(f^m)' + p\eta f\}d\eta + (p+q)\int_0^\infty \phi\, f\, d\eta = 0$$
$$for\ all\ \phi \in C_0^1[0,\infty).$$

Now, we establish the following results.

(i) Let $f_0 > 0$, then problem (1.7), (1.8) has a weak solution with compact support if and only if $p \geq 0$ and $2p + q > 0$. This solution is unique.

(ii) Let $f_0 = 0$ then problem (1.7), (1.8) has a non-trivial weak solution with compact support if and only if $p > 0, 2p + q = 0$.

Suppose if and only if $p > 0, \ \ 2p + q = 0$

In this case, there exist a one parameter family of such solutions.

II. THE METHOD

Let f be a weak solution of problem (1.7), (1.8) with compact support in $[0,\infty)$.

$\Rightarrow f > 0$ in the right neighborhood of $\eta = 0$. i.e. there exists a number $a > 0$ such that $f > 0$ on $(0, a), f = 0$ on $[a, \infty)$.

It was shown in [3] that in a neighborhood of any point where $f > 0, f$ is classical solution of equation (1.7). Thus, we shall be concerned with proving the existence and uniqueness of a classical positive solution o (1.7) on $(0, a)$ which satisfies the boundary conditions

$$f(0) = f_0 \tag{2.1}$$
$$f(a) = 0, \quad (f^m)'(a) = 0 \tag{2.2}$$

The condition at $\eta = a$ follows from the requirement that f and $(f^m)'$ are continuous on $(0,\infty)$.

Before turning to the existence, we obtain a preliminary non-existence result.

LEMMA 1

The existence of non-trivial weak solution of equation (1.7) with compact support implies one of the following propositions.

(i) $p > 0$ or

(ii) $p = 0$ and $q > 0$

PROOF:

Suppose, f is a non-trivial weak solution of (1.7) with compact support. Then there exists $a > 0$, such that

$f > 0$ in $(a - \varepsilon, a)$ for some $\varepsilon > 0$ and

$f = 0$ in $[a, \infty)$.

Thus in $(a - \varepsilon, a), f$ satisfies (1.7) and at $\eta = a$, f satisfies (2.2). Integration of (1.7) from $\eta \in (a - \varepsilon, a)$ to a yields

$$-(f^m)'(\eta) = p\eta\, f(\eta) + (p+q)\int_\eta^a f(\xi)\, d\xi \tag{2.3}$$

In view of the continuity of f and $(f^m)'$ it is possible to find $\eta_0 \in (a-\varepsilon, a)$ such that $f'(\eta_0) < 0$

Hence, p and $(p+q)$ cannot both be less than zero.

Thus, if $p = 0, q$ must be positive. Now, suppose that $p < 0$. Then by (2.3), $p + q > 0$ and hence $q > 0$. It follows from (1.7) that f cannot have a maximum in $(a-\varepsilon, a)$ and hence $f' < 0$ on $(a-\varepsilon, a)$. Therefore, (2.3) implies

$$-mf^{m-2}(\eta)f'(\eta) - p\eta \leq (p+q)(a-\eta) \tag{2.4}$$

for all $\eta \in (a-\varepsilon, a)$. If we now let $\eta \to a$, we obtain a contradiction.
Hence, $p > 0$.

III. SOLUTION NEAR $\eta = a$

Let a be an arbitrary positive number. It is clear from Lemma 1, that a necessary condition for the existence for a positive solution of problem (1.7), (2.2) in the neighbourhood of $\eta = a$ is that either $p > 0$ or $p = 0$ and $q > 0$. Now, we show that this condition is also sufficient. For that, let $p = 0$ and $q > 0$. Then we can solve problem (1.7), (2.1),(2.2) uniquely and

$$f(\eta, a) = \left\{\frac{q(m-1)^2}{2m(m+1)}(a-\eta)^2\right\}^{\frac{1}{m-1}} \qquad 0 < \eta < a \tag{3.1}$$

is an unique solution of problem (1.7), (2.2). Because $f(0, a)$ is continuous, monotonically increasing function of a such that $f(0,0) = 0$ and $f(0, \infty) = \infty$, the equation $f(0, a) = f_0$ is uniquely solvable for $f_0 \geq 0$. Let $a(f_0)$ be its solution, then $f = f(\eta, a(f_0))$ is an unique solution of problem (1.7), (2.1), (2.2).

Now, consider the case when $p > 0$. First we prove the following lemma.

LEMMA 2

Let $b \in (0, a)$ and let f be a positive solution of the problem (1.7), (2.2) on $[b, a)$.

(i) If $p + q \geq 0$ then $f'(\eta) < 0$ on $[b, a)$.

(ii) If $p + q < 0$, and there exist an $\eta_0 \in [b, a)$ such that $f'(\eta_0) = 0$ then f has a maximum at η_0 and $\eta_0 < \left[\frac{p+q}{q}\right] a$.

If f is a positive solution of (1.7), (2.2) on $[0, a)$ then

(i) $p + q > 0, f'(0) < 0$

(ii) $p + q = 0, f'(0) = 0$

(iii) $p + q < 0, f'(0) > 0$

PROOF

Integrating of (1.7) from $\eta \in [b, a)$ to a yields (2.3). If $p + q \geq 0$, this implies that $(f^m)'(\eta) < 0$ and hence $f'(\eta) < 0$ on $[b, a)$.

If $p + q < 0$, we note that $q < 0$ and hence $f'(\eta_0) = 0 \Rightarrow f''(\eta_0) < 0$.

If follows that, f has maximum at $\eta = \eta_0$ and $f'(\eta) < 0$ on (η_0, a).

To estimate η_0, we set $\eta = \eta_0$ in (2.3) and using the fact that $f'(\eta_0) < 0$ on (η_0, a) we obtain,

$$0 = p\eta_0 f(\eta_0) + (p+q) \int_{\eta_0}^{a} f(\xi)\, d\xi$$

$$> p\eta_0\, f(\eta_0) + (p+q) \int_{\eta_0}^{a} f(\eta_0)\, d\xi$$

Hence, $p\eta_0 + (p+q)(a-\eta_0) < 0$ or $(p+q)a - q\eta_0 < 0$.

Recalling that, $q > 0$, we obtain upper bound for η_0 viz.

$$\eta_0 < \left[\frac{p+q}{q}\right] a$$

Finally, if we set $\eta = 0$, (2.3) yields,

$$-(f^m)'\,(0) = (p+q) \int_0^a f(\xi)\, d\xi$$

from which sign of $f'(0)$ follows. Now, we procced for existence.

LEMMA 3

Let $p > 0$ and let q be arbitrary. Then given any $a > 0$, there exists an $\varepsilon > 0$ such that problem (1.7), (2.2) has a unique positive solution in $(a-\varepsilon, a)$

PROOF

As in [3], we reduce the problem to that of establishing the local existence of solution of an equivalent integral equation. To derive this let f be a positive solution in $(a-\varepsilon, a)$ for some $\varepsilon > 0$.

By lemma 2, it is possible to choose an $\varepsilon > 0$ such that $f' < 0$ in $(a-\varepsilon, a)$. Therefore, consider an inverse function $\eta = \sigma(f)$.

Rewriting (2.3) as,

$$(f^m)'(\eta) = q\eta\, f(\eta) - (p+q) \int_{\eta}^{a} f(\xi) d\xi$$

Hence, $\sigma(f)$ satisfies the integro-differential equation,

$$\frac{d\sigma}{df} = \frac{m\, f^{m-1}}{q\, f\, \sigma(f) - (p+q) \int_0^f \sigma(\phi) d\phi}$$

Integrating from 0 to f yields,

$$\sigma(f) - a = m \int_0^f \frac{\phi^{m-1}\, d\phi}{q\, \phi\, \sigma(\phi) - (p+q) \int_0^\phi \sigma(\Psi) d\Psi}$$

or introducing $\tau(f) = 1 - a^{-1}\sigma\,(f)$ then,

$$\tau(f) = \frac{m}{a^2} \int_0^f \frac{\phi^{m-1}\, d\phi}{q\, \phi + q\, \phi\, \tau(\phi) - (p+q) \int_0^\phi \tau(\Psi) d\Psi} \tag{3.2}$$

Now, we prove that, (3.2) has a unique positive solution in a right neighborhood of $f = 0$.

Let $\lambda > 0$ and let X be a function $\tau(f)$ defined on $[0.\gamma]$, such that

$$0 \le \tau(f) \le \rho = \frac{p}{2(|q| + |p+q|)}$$

We denote by $||.||$ the supremum norm on X, then X is a complete metric space. We define the operator,

$$M(\tau)(f) = \frac{m}{a^2} \int_0^f \frac{\phi^{m-1} d\phi}{p\phi + q\phi\, \tau(\phi) - (p+q) \int_0^\phi \tau(\psi) d\psi}$$

Let $\tau \in X$ then,

$$p\phi + q\phi\tau(\phi) - (p+q)\int_0^{\phi}\tau(\psi)d\psi$$
$$\geq \{p - (|q| + |p+q|)||\tau||\}\cdot\phi$$
$$\geq \frac{1}{2}p\phi$$

Hence, $M(\tau)(f) \leq \frac{m}{a^2}\int_0^f \frac{\phi^{m-2}}{\frac{1}{2}p\phi}d\phi \leq \frac{2m}{(m-1)pa^2}\ \gamma^{m-1}$

Thus, $M(\tau)$ is well defined on the whole of X. Thus,

$M(\tau)\colon [0,\gamma] \to R$ is non-negative and continuous and moreover there exists $\gamma_0 > 0$ such that if $\gamma < \gamma_0$ and $\tau \in X, ||M(\tau)|| \leq \rho$.

Thus, if $\gamma \leq \gamma_0$ then, M maps X into X.

Let $\tau_1, \tau_2 \in X$ and let $\gamma \leq \gamma_0$ then,

$$||M(\tau_1) - M(\tau_2)||$$
$$\leq \frac{4m}{a^2p^2}\int_0^f \phi^{m-3}\left[|q|\phi||\tau_1-\tau_2|| + |p+q|\int_0^{\phi}||\tau_1-\tau_2||\,d\psi\right]d\phi$$
$$\leq \frac{4m}{(m-1)a^2p^2}\,(|q| + |p+q|)||\tau_1-\tau_2||\cdot\gamma^{m-1}$$

Hence, there exists $\gamma_1 \in (0,\gamma_0]$ such that if $\gamma \leq \gamma_1$, M is a contraction on X. thus, by Banach-Cacciopolo contraction mapping principle [7, p.404], M has a unique fixed point in X and equation (3.2) has a unique solution.

IV. BACKWARD CONTINUATION

Let $a > 0$ and $f(\eta)$ be the solution of (1.7), (2.2) we constructed in the previous section. Then f is defined and positive in a left neighborhood of $\eta = a$. Now, we continue f backwards as a function of η. By the standard theory [7], this can be done uniquely so long as f remains positive and bounded. Now, there are three possibilities.

(a) $f(\eta) \to \infty$ as η decreases to some $\eta_1 \in [0,a)$.

(b) $f(\eta)$ can be continued back to $\eta = 0$.

(c) $f(\eta) \to 0$ as η decreases to some $\eta_2 \in [0,a)$.

Now, we try to rule out possibility (a).

LEMMA 4

Let $b \in \{0,a)$, and let f be a positive solution of problem (1.7), (3.1) on (b,a).

Then, if $p > 0$,

$$\sup_{(b,a)} f(\eta) \leq \left[\frac{m-1}{2m}a^2\max\{p, 2p+q\}\right]^{\frac{1}{m-1}}$$

PROOF

(i) Let $p+q \geq 0$, then by Lemma 2, $f' < 0$ on (b, a). Using in (2.4), we get,
$-m\, f^{m-2}(\eta) f'(\eta) \leq (p+q)a - q\eta \qquad b \leq \eta \leq a.$

Integration from η to a yields,

$$\frac{m}{m-1}\, f^{m-1}(\eta) \leq (a-\eta)\left[pa + \frac{1}{2}q(a-\eta)\right],\ b \leq \eta \leq a \tag{4.1}$$

and hence,

$$\sup_{(b,a)} \frac{m}{m-1}\, f^{m-1}(\eta) \leq \frac{1}{2}(2p+q)a^2 \tag{4.2}$$

(ii) Let $p+q < 0$. Then, it follows from (2.3), that,
$-mf^{m-1}(\eta) f'(\eta) \leq p\, \eta\, f(\eta)$

If we divide by $f(\eta)$ and integrate from η to a, we get,

$$\frac{m}{m-1}\, f^{m-1}(\eta) \leq \frac{1}{2}p\,(a^2-\eta^2), \qquad b \leq \eta \leq a \tag{4.3}$$

Thus,

$$\sup_{(b,a)} \frac{m}{m-1} f^{m-1}(\eta) \leq \frac{1}{2}pa^2 \tag{4.4}$$

Because the bound of Lemma 4 is uniform in b, $f(\eta)$ can never become unbounded as η decreases.

The estimates (4.1) and (4.3) provide upper bounds for $f(\eta)$ which also tends to zero as $\eta \to a$. Lower bounds can be derived in exactly the same way, one finds

(i) If $p+q \geq 0$.

$$\frac{m}{m-1}\, f^{m-1}(\eta) \geq \frac{1}{2}p(a^2-\eta^2),\ \ b \leq \eta \leq a \tag{4.5}$$

(ii) If $p+q < 0$.

$$\frac{m}{m-1}\, f^{m-1}(\eta) \geq \left\{pa + \frac{1}{2}q(a-\eta)\right\}(a-\eta), \tag{4.6}$$
$$max.(b, \eta_0) \leq \eta \leq a.$$
$$\geq \frac{1}{2}(2p+q)(a^2-\eta^2).$$

The following lemma distinguishes between the possibilities (b) and (c).

LEMMA 5

Let f be the positive solution of problem (1.7),(2.2) in a left neighbourhood of $\eta = a$. Assume that $p > 0$, then,

(i) If $(2p+q) > 0, f(\eta) > 0$ on $[0, a)$.
(ii) If $(2p+1) = 0, f(\eta) > 0$ on $(0, a)$ and $f(0) = 0$.
(iii) If $(2p+q) < 0$, there exists on $\eta^* \in (0, a)$ such that $f(\eta^*) > 0$ on (η^*, a) and $f(\eta^*) = 0$.

PROOF

Integrating of (2.3) from η to a yields the following integral equation for f:

$$(f^m)(\eta) = p\eta \int_\eta^a f(\xi)d\xi + (2p+q)\int_\eta^a (\xi-\eta)f(\xi)d\xi \tag{4.7}$$

Now, suppose $2p+q>0$, then by the previous Lemma we may continue $f(\eta)$ back to $\eta=0$, and $f(0)>0$. However, using the bounds for f, we can actually give upper and lower bounds for $f(0)$. This can be done by the following proposition and for that we define the quantities,

$$\lambda = \frac{2p+q}{p},\ \ \mu = 1-\left[\frac{p+q}{p}\right]^2,\ A = \left[\frac{m-1}{2m}pa^2\right]^{\frac{1}{m-1}}$$

PROPOSITION 1

Let $p>0$, and $2p+q>0$, then,

(i) If $p+q \geq 0$ $(\lambda \geq 1)$
$\lambda^{\frac{1}{m}}A \leq f(0) \leq \lambda^{\frac{1}{m-1}}A$
(ii) If $p+q \leq 0$ $(0<\lambda \leq 1)$

$(\mu\,\lambda)^{\frac{1}{m-1}}A \leq f(0) \leq \lambda^{\frac{1}{m}}A$

Both estimates are sharp for $p+q=0$

PROOF

(i) The upper bound follows at once from (4.1). To obtain lower bound, we use (4.6)

in (4.7),

$$(f^m)(0) = (2p+q)\int_0^a f(\xi)d\xi \tag{4.8}$$

Result follows after an elementary computation,
(ii) In this case, we only have a bound for f on $[\eta_0, a)$, where η_0 is the value for η for

which f reaches to maximum. By (4.3) and (4.6),

$$\lambda^{\frac{1}{m-1}}A\left[1-\frac{\eta^2}{a^2}\right]^{\frac{1}{m-1}} \leq f(\eta) \leq A\left[1-\frac{\eta^2}{a^2}\right]^{\frac{1}{m-1}},\quad \eta_0 \leq \eta \leq a \tag{4.9}$$

However $f(\eta) \leq f(\eta_0)$ on $[0,\eta_0]$ and therefore (4.9) holds for $0 \leq \eta \leq a$. Using (4.9) in (4.8), we get desired upper bound.

To obtain lower bound, we note by (4.8), that

$$(f^m)(0) \geq (2p+q)\int_{a*}^a \xi\, f(\xi)d\xi \tag{4.10}$$
$$where\ a^* = \frac{p+q}{p}a.$$

Because by Lemma 2, $\eta_0 \leq a^*$ we can use (4.9) in (4.10) to estimate $f(0)$, we conclude this with a result about the dependence of f on the choice of a^*.

PROPOSITION 2

Let $p > 0$ and $2p + q \geq 0$. Suppose $f(\eta, a_1)$ and $f(\eta, a_2)$ are solutions of problem (1.7), (2.2) on $(0, a_1)$ and $(0, a_2)$ respectively. Then if $a_1 > a_2, f(\eta, a_1) > f(\eta, a_2)$ everywhere on $(0, a_2)$.

PROOF

We denote $f(\eta, a_1)$ by $f_i(\eta)\ for\ i = 1,2$.

Suppose proposition is not true, therefore there exists an $\bar{\eta} \in (0, a_2)$ such that $f_1(\bar{\eta}) = f_2(\bar{\eta})$ and $f_1(\eta) > f_2(\eta)$ on $(\bar{\eta}, a_2)$.

It follows from (4.7) that for $i = 1,2$

$$f_i^m(\bar{\eta}) = p\,\bar{\eta} \int_{\bar{\eta}}^{a_i} f_i(\xi)\ d\xi\ + (2p + q) \int_{\bar{\eta}}^{a_i} (\xi - \bar{\eta}) f_i(\xi) d\xi$$

$$\text{Here,}\quad p\,\bar{\eta} \int_{a_2}^{a_1} f_1(\xi)\ d\xi\ + (2p + q) \int_{a_2}^{a_1} (\xi - \eta) f_i(\xi) d\xi$$

$$+p\,\bar{\eta} \int_{\bar{\eta}}^{a_2} [f_1(\xi) - f_2(\xi)]\ d\xi\ + (2p + q) \int_{\bar{\eta}}^{a_2} (\xi - \bar{\eta}) [f_1(\xi) - f_2(\xi)] d\xi\ = 0$$

The second and the fourth term of this expression are non-negative, while the other two are positive, therefore we have a contradiction.

V. MAIN RESULT

We now begin by proving existence and uniqueness of the solution of problem (1.7), (2.1), (2.2) which is positive on $(0, a)$. By Lemma 1, a necessary condition for the existence of such a solution is that $p \geq 0$.

Let $p > 0$. Then by Lemma 3, for each $a > 0$, there exists a unique positive solution $f(\eta, a)$ of (1.7), (2.2) in a left neighborhood of $\eta = a$. By Lemma 5, this solution can be continued back to $\eta = 0$ if and only if $2p + q \geq 0$. Thus, the boundary condition at $\eta = 0$ is satisfied if we can find an $a > 0$ such that

$$f(0, a) = f_0 \qquad (5.1)$$

If only one such a exists, the solution is unique.
Here two cases arise

(i) $f_0 = 0$ Then, by Lemma 5, equation (4.1) can only be satisfied if $2p + q = 0$. Moreover, (5.1) is then satisfied for any $a > 0$.
(ii) $f_0 > 0$. Then, by Lemma 5, a necessary condition for (5.1) to have solution is that $2p + q > 0$. To prove that, it is sufficient we use observation due to Bareblatt [8].

Let $f(\eta, a)$ be a solution problem (1.7), (2.2) on $(0, a)$. Thus, choosing $\mu = a^{-1}$,

$$f(0, a) = a^{\frac{2}{m-1}} f(0,1)$$

Therefore (5.1) can be written as

$$a^{\frac{2}{m-1}} f(0,1) = f_0 \tag{5.2}$$

Because $2p + q > 0, f(0,1) > 0$. It follows that for each $f_0 > 0$ equation (5.2) has a unique solution $a\ (f_0)$. The function $f\ (\eta, a(f_0))$ now satisfies (1.7),(2.1), (2.2). In view of the uniqueness of $a(f_0)$ it is the only function which does so. Remembering the solution we constructed for $p = 0$, we have proved the following results.

THEOREM 1

(i) Let $f_0 > 0$, then there exists a unique $a > 0$ and a unique solution of problem (1.7), (2.1), (2.2) which is positive on $(0, a)$ if and only if $p \geq 0$ and $2p + q > 0$.

(ii) Let $f_0 = 0$. Then for every $a > 0$ there exists a unique solution of problem (1.7), (2.1), (2.2) which is positive on $(0, a)$ if and only if $p > 0$ and $2p + q = 0$.

Therefore, it is easy to see that

$$f(\eta) = \begin{cases} f(\eta, a) & 0 \leq \eta < a \\ 0 & a \leq \eta < \infty \end{cases}$$

is a weak solution of (1.7) which satisfies the boundary condition (1.8). Hence, we show that if $f_0 > 0$, this is the only solution of problem (1.7), (1.8) with compact support and that if $f_0 = 0$ this is the only family of non-trivial solution of problem (1.7), (1.8) with compact support.

Let $f(\eta)$ be a weak solution of the problem (1.7), (1.8) with compact support. Therefore, it follows from Lemma 5, that if $f_0 > 0$, problem (1.7), (1.8) only has such a solution if $2p + q > 0$ and it is of the form

$f(\eta) > 0$ on $[0, a)$.
$f(\eta) = 0$ on $[a, \infty)$.

for some $a > 0$. That is, f must be of the type discussed above, and by Theorem 1, there exists only one such solution.

If $f_0 = 0$, besides the family of solution discussed above, one might expect non-trivial solution which are zero on a disconnected subset of $(0, \infty)$. We now prove that such solution cannot exist.

Let f be a weak solution such that $f > 0$ on (a_2, a_1), where $0 < a_1 < a_2 < \infty$ and $f = 0$ at $\eta = a_1$ and $\eta = a_2$. Then, for f to be a weak solution of (1.7), we require,

$$f(a_i) = 0, \ \ (f^m)'(a_i) = 0 \qquad\qquad i = 1,2.$$

On $(a_1, a_2), f$ is a classical solution of (1.7) and hence integration of (1.7) from a_1 to a_2 yields

$$0 = (p + q) \int_{a_1}^{a_2} f(\xi) d\xi$$

Because $p + q = (2p + q) - p < 0$ and $f > 0$ on (a_1, a_2) we arrive at a contradiction .

It follows that if $f_0 = 0$, any weak solution of problem (1.7),(1.8) with compact support must belong to the family of solution discussed above. Thus, we have proved the following theorem.

Theorem 2

(i) Let $f_0 > 0$. Then there exists a unique weak solution with compact support of problem (1.7), (1.8) if and only if $p \geq 0$ and $2p + q > 0$

(ii) Let $f_0 = 0$. Then there exists a non-trivial weak solution with compact support of (1.7), (1.8) if and only if $p > 0$ and $2p + q = 0$. For solution f with the property $f > 0$ on $(0, a)$ and $f = 0$ on $[a, \infty)$.

REFERENCES

1. A. P. Verma, Can. J. Phy. 47, 1969, 2519-2524
2. O. A. Oleinik, A. S. Kalashnikov and CHZ HOU Yui-Lin, 'The Cauchy problem and boundary problems for equations of the type unsteady filteration' , Izv, Akad, SSSR Ser. Mat. 22, 1958, 667-704.
3. F. V. Atkinson and L. A. Peletier, 'Similarity profiles of flows through Porous Medium', Arch. Rational Mech. Anal 42, 1961, 369-379.
4. F. V. Atkinson and L. A. Peletier. 'Similarity solutions of the non-linear diffusion equation', Arch. Rational Mech. Anal. 54, 1974, 373-392.
5. L.F. Shampine, 'Concentration dependent diffusion', Quart, Appl. Maths. 30, 1973, 441-452.
6. L. F. Shampine, 'Concentration dependent diffusion-II singular problems', Quart, Appl. Maths. 31, 1973, 287-452.
7. P. Hartman, 'Ordinary differential Equation', John Wiley and sons Inc. New York, 1964.
8. G. L. Barenblatt, 'On some unsteady motions of a liquids and a gas in a porous medium', Prinkl, Mat. Mech. 16, 1952, 67-68.

Cite this Article

Pandya Parth M, Dr. Gajendra Purohit, Dr. P. H. Bhathawala, "Existence and Uniqueness Theorem for the Flow of two Immiscible Fluids Through Porous Media with Decreasing Exponential Saturation", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 7 Issue 6, pp. 472-482, November-December 2020.

Journal URL : https://ijsrst.com/IJSRST218141