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Study on Similarity, Metrically Equivalence Operators

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ABSTRACT

This paper has been devoted to the study on similarity, metrically equivalence operators. In this paper we have studied lies in the inverses and adjoints of operators, in connection with similarity and obtained some results. In continuation we have studied invertible operators and conjectures on similarity and give some results in the form of theorem. We also study the concept of metrically equivalent operators and obtained some results involving this concept in the form of theorems.

Keywords : Similarity, metrically equivalence operators, inverses and adjoints of operators, hyponormal operator, conjectures.

I. INTRODUCTION

Many researchers De Prima [3], Halmos [5], Choi's [2], Radjavi and Williams [6], Singh,U.N [9], Williams [4] are studied and obtained some results on similarity, metrically equivalent operators. In this paper, we have study lies in the inverses and adjoints of operators, in connection with similarity. We also study the concept of metrically equivalent operators and obtained some results involving this concept in the form of theorems. We define some definitions:

Definition (1.1):
Two operators S and T in $B(H)$ are said to be metrically equivalent if $S^*S = T^*T$.

Definition (1.2):
An operator T is said to be
(i) a numerical contraction if $w(T) \leq 1$.
(ii) normaloid if $r(T) = ||T||$ or $||T^n|| = ||T||^n$ for positive integer n.
(iii) Convexoid if conv $\sigma(T) = \overline{W(T)}$, where "conv" denotes the convex hull i.e. the intersection of all convex sets containing $\sigma(T)$ and $\overline{W(T)}$ denotes the closure of $W(T)$.
(iv) Spectraloid if $r(T) = w(T)$. ([7]) .

Definition (1.3):
Two operators M and N are said to be congruent if there exists an invertible operator T satisfying $M = TNT^*$.
We have the following result in the form of theorem which gives the information about the spectrum of such operators.

Theorem (1.1):
If T be an invertible operator satisfying $T = S^{-1}T^{*-1}S$, Then there exists a congruence C of S such that $\sigma(C) = \sigma(S^{*-1})$.

Proof:
Using adjoints and inverses operators,
(1.1) $T = S^{-1}T^{*-1}S \Rightarrow S^{*-1}T^{*-1} = TS^{*-1}$
or $S^{*-2}T^{*-1} = S^{*-1}T\, S^{*-1}$
or $S^{*-2} = S^{*-1}TS^{*-1}T^*$
or (1.2) $S^{*-1} = S^{*-1}TS^{*-1}T^*S^*$.
Puting $C = TS^{*-1}T^*$ in (1.2), we get
(1.3) $S^{*-1} = S^{*-1}CS^*$.
since C is similar to S^{*-1},
Hence the proof.

II. INVERTIBLE OPERATORS AND CONJECTURES ON SIMILARITY

Choi [2], Radjavi and Williams [6] studied conjectures and give some results. Choi [2] proves that an invertible operator T is normal if and only if co-square of T is unitary. We obtain a characterization of a hyponormal operator in the following:

Hyponormality of T implies that

(2.1) $(T^{*-1}T)(T^{*-1}T)^{*} = T^{*-1}T\,T^{*}T^{-1} \le T^{*-1}T^{*}T\,T^{-1} = I.$

Hence $T^{*-1}T$ is a contraction,

and conversely,

If $T^{*-1}T$ is a contraction then T is hyponormal.

After these observation we have take some results in the form of theorems:

Theorem (2.1):

An invertible operator T is hyponormal if and only if co-square of T is a contraction.

Choi's [2] concept of co-normal operator suggests that one can introduce the following notion.

Definition (2.1):

An invertible operator T is said to be hypoconormal if

(2.2) $(T^{*-1}T)(T^{*-1}T)^{*} \le (T^{*-1}T^{-1})(T^{*-1}T^{-1})^{*}.$

Hypoconormal operators are characterized in the following:

Theorem (2.2):

An invertible operator T is hypoconormal if and only if $T^{*}T \le (T^{*}T)^{-1}$.

Proof: From (2.2), we have

$$(T^{*-1}T)(T^{*-1}T)^{*} \le (T^{*-1}T^{-1})(T^{*-1}T^{-1})^{*}$$
$$\Leftrightarrow T^{*-1}T\,T^{*}T^{-1} \le T^{*-1}T^{-1}T^{*-1}T^{-1}$$
$$\Leftrightarrow TT^{*} \le T^{-1}T^{*-1} = (T^{*}T)^{-1},$$

Hence the results.

Now Choi [2] some results we have take in the form of theorems:

Theorem (2.3):

T is conormal if and only if

$$(T^{*-1}T)(T^{*-1}T)^{*} = (T^{*-1}T^{-1})(T^{*-1}T^{-1})^{*}.$$

Also, from theorem (2.2) we see that T is hypoconormal if and only if T^2 is a contraction.

Theorem (2.4):

Co-square of an operator T is unitary if and only if T commutes with the co-square.

Theorem (2.5):

T is normal if and only if T^2 commutes with its co-square.

We now the following (Choi's Conjecture [2]):

Theorem (2.6):

Let T in $B(H)$ be invertible. Then T^* is similar to T^{-1} if and only if T is a co-square.

Proof:

If T is a co-square, then by definition there exists an invertible operator S satisfying

(2.3) $T = S^{*-1}S$,

Then (2.4) $T^* = S^*S^{-1}$,

From (2,3), we have

(2.5) $T^{-1} = S^{-1}S^*$,

or $T^{-1} = S^{-1}S^*S^{-1}S = S^{-1}T^*S$,

i.e., (2.6) $T^{-1} = S^{-1}T^*S$

(2.6) implies that $T^*$is similar to T^{-1}.

and conversely.

Remark:

(i) If T is similar to T^*, then some similarity of T is unitarily equivalent to its adjoint.

(ii) Two similar normal operators are unitarily equivalent and adjoint of a normal operator is again normal.

The above conjecture is known to be solved for normal operators.

From these observations we have take result in the form of proposition:

Proposition 2.1:

Let T be similar to T^* with a normal operator$S$. Then some similarity of T is unitarily equivalent to its adjoint.

Proof:

Let the operator

(2.7) $T^* = STS^{-1}$

with normal S,

and (2.8) $S = UP$

be its polar decomposition,

then U and P commute i.e.

(2.9) $UP = PU$,

and

(2.10) $T^* = UPTP^{-1}U^* = PUTU^*P^{-1}$.

Now let R be the positive square-root of P, then $R^2 = P$,

and R is also commutes with U i.e., $RU = UR$, form (2.10) we have

(2.11) $T^* = R^2UTU^*R^{-2}$

From (2.11), we have

(2.12) $R^{-1}T^*R = RUTU^*R^{-1}$

Or (2.13) $R^{-1}T^*R = URTR^{-1}U^*$

Thus $T = RTR^{-1}$ is unitarily equivalent to its adjoint $R^{-1}T^*R = T^*$.

Hence complete the proof.

III. METRICALLY EQUIVALENT OPERATORS:

We have study the concept of metrically equivalent operators. Berberian [1], Sheth [8]. gives some results involving this concept in the form of theorems:

Theorem (3.1):

Let an operator S be a normaloid and T be any other operator with $r(S) \leq r(T)$, If S is metrically equivalent to T, then T is also normaloid.

Proof: From our hypothesis we have $S^*S = T^*T$, i.e.

$$||S||^2 = ||T||^2,$$

Since S is normaloid, we have

$$||S^2|| = ||T||^2.$$

Using the usual spectral inequality, we have

$$r(T)^2 = r(T^2) \leq ||T^2|| \leq ||T||^2 = ||S||^2 = r(S)^2.$$

or $$r(T) \leq r(S).$$

This, combined with the hypothesis, yields the desired conclusion.

It may also be observed that an operator metrically equivalent to an partial isometry is an partial isometry.

Theorem (3.2):

Let an operator S be metrically equivalent to T, then T is normal if and only if there exists a unitary operator U which commutes with $(S^*S)^{1/2}$.

Proof:

From normality of the operator T implies that

$$T^* = UT$$

with a unitary operator U.

Clearly $S^*S = T^*T$ yields

Now we have

$$S^*S = UT^2,$$

or $$T^2 = U^*S^*S,$$

the normal nature of T is inherited by T^2, and U^*S^*S is the polar decomposition of T^2 in which U^* and S^*S commute and hence U commutes with $(S^*S)^{1/2}$.

Conversely, if U commutes with $(S^*S)^{1/2}$ then U also commutes with $(T^*T)^{1/2}$.

Putting $T = UP$ with $P = (T^*T)^{1/2}$,

it is obvious that T is normal and we are done.

Berberian [1] we have take the following results in the form of theorem for metrically equivalence:

Theorem (3.3):

If the operators S and T are metrically equivalent normal operators then there exists a unitary operator U such that $S = UT$.

Theorem (3.4):

An operator T is hyponormal if and only if there exists a contraction V such that $T^* = VT$.

Theorem (3.5):

Let an operator T be a hyponormal operator which is metrically equivalent to an operator S for which $S^* = US$ with U an isometry. Then there exists a contraction V satisfying $S^2 = VT^2$.

Proof:

From theorem (2.4), we see that for a hyponormal operator ,

$$T^* = V_1 T$$

with a contraction V_1.

Hence (2.14) $T^* T = V_1 T^2$.

Also $S^* = US$

or (2.15) $S^* S = US^2$.

Thus T and S metrically equivalent and from (2.14) and (2.15) shows that

$$V_1 T^2 = US^2,$$

with an isometry U,

or $S^2 = U^* V_1 T^2$.

Putting $V = U^* V_1$, we get the desired conclusion.

IV. CONCLUSION

This paper has been divided into three sections of which the first section is introductory and the second section deals with invertible operators and conjectures on similarity. In this section, we have find some results in the form of theorem. Also we obtain a characterization of a hyponormal operator and find some results in the form of theorem. In the third section we have studied the concept of metrically equivalent operators and gives some results involving this concept in the form of theorems.

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