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An Analysis of a Two Dimensional Continuous Non-Linear Dynamical Systems

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ABSTRACT

The prediction of physical phenomenon commonly observed in nature has been a tough challenge before the scientists and mathematicians all over the world. A careful mathematical modeling of such events has helped us to predict the physical state of a system given the current state. Non-linear dynamical systems like mass-spring systems, electrical circuits, chemical reactions, predator-prey models, Lorenz equations, damped driven pendulum, Van der Pol oscillator, and many more have been studied by many mathematicians and physicists and the strange behavior, so called chaos, has been observed in such systems. As an example of a chaotic dynamical system, we have considered the Duffing oscillator, which is an extremely forced and damped oscillator. In this paper, we have analyzed the dynamics of the Duffing oscillator. We have constructed the differential equation of the motion of the Duffing oscillator, obtained its critical points and classified them in reference to their stability. Also, we have obtained the solutions for different initial conditions and different ranges of parameters and concluded that the system exhibits chaotic behavior.

Keywords: Dynamical systems, nonlinear oscillators, equilibrium points , period doubling, chaos.

Mathematics Subject Classification: 37, 37C, 37C05, 37C10, 37C20, 37C25, 37C27, 37C35, 37D, 37G.

I. INTRODUCTION

First we will have a brief discussion and references about the general notions and definitions we will need to understand that come across this paper. Among many definitions of a dynamical system, we prefer a general definition as suggested by Edward R. Scheinerman. [10]

1.1 Dynamical System [10]:

A *dynamical system* is specified by a state vector $X \in R^n$, which is a list of numbers which may change as time progresses and a function $F : R^n \to R^n$ which describes how the system evolves over time. A continuous time dynamical systems has a state vector $X(t) \in R^n$ and we are given a function $F : R^n \to R^n$ which specifies how quickly each component of $X(t)$ is changing, *i.e.*, $X'(t) = F(X(t))$, or in brief notation, $X' = F(X)$, which is a system of differential equations.

It is well known that many physical phenomena can be mathematically modeled in terms of differential equations and the difference equations and the long term effects can be studied over time. Differential equations can be used to describe the motions of objects like satellites, water molecules in a stream, waves on strings and surfaces, etc. In this section we will take a review of some basic terminology associated with a system of differential equations.

1.2 System of Differential Equations [9]:

Let $x_1,\ x_2, \dots, x_n$ be differentiable functions of a variable t, usually called as time, on an interval I of the real numbers. Let $f_1,\ f_2, \dots, f_n$ be all functions of $x_1,\ x_2, \dots, x_n$ and t. Then the set of n differential equations

$$
\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1,\ x_2, \dots, x_n, \mathrm{t}),\\
\frac{dx_2}{dt} &= f_2(x_1,\ x_2, \dots, x_n, t),\\
&.\\
&. \qquad\qquad\qquad\qquad (1)\\
&.\\
\frac{dx_n}{dt} &= f_n(x_1,\ x_2, \dots, x_n, \mathrm{t})
\end{aligned}
$$

is called as a *system of differential equations*. This system can also be expressed as $X' = F(X, t)$, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, X' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \text{ and } F = (f_1,\ f_2, \dots, f_n).$$

The system $X' = F(X, t)$, where F can depend on the independent variable t is called as a non-autonomous system. Any non-autonomous system (1) with $X \in R^n$ can be written as an autonomous system

$$X' = F(X) \qquad (2)$$

with $X \in R^{n+1}$ simply by letting $x_{n+1} = t$ and $x_{n+1}' = 1$. The fundamental theory for the systems (1) and (2) does not differ significantly.

1.3 Phase-Plane Analysis[11]:

If $X: I \to R^n$ is defined by $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, and if $X(t)$ satisfies the system (1), then $X(t)$ is said to be a solution of the system (1). If $t_0 \in R$ and X is s solution for all $t \geq t_0$, then $X(t_0)$ is an initial condition of a solution X. As $x_1,\ x_2, \dots, x_n$ are functions of the variable t, it follows that as t increases, $X(t)$ traces a curve in R^n called as the *trajectory* or the *orbit* and in this case, the space R^n is called as the *phase space* of the system. The phase space is completely filled with trajectories since each point $X(t_0)$ can serve as an initial point. The system $X' = F(X)$ is said to be a *linear system* if the function F is linear. In this case, the system can be expressed as $X' = A.X$, where A is an $n \times n$ matrix. The function F is also called as a *vector field.* The vector field always dictates the velocity vector $X'$for each X. A picture which shows all qualitatively different trajectories of the system is called as a *phase portrait.* A second order differential equation which can be expressed as a system of two differential equations can be treated as a vector field on a plane and hence called as a *phase plane*. The general form of a vector field over the plane is

$$x_1{}' = f_1(x_1,\ x_2),$$
$$x_2{}' = f_2(x_1,\ x_2),$$

which can be compactly written in vector notations as $X' = F(X)$,where $X = (x_1,\ x_2)$ and $F(X) = (f_1(X),\ f_2(X))$.

For non-linear systems, it is quite difficult to obtain the trajectories by analytical methods and though the trajectories are obtained by explicit formulas, they are too complicated to provide some information about the solution. Hence qualitative behaviors of the trajectories obtained by numerical solution methods are often studied. To obtain a phase portrait, we plot the variable x_1 against the variable x_2 and study the qualitative behavior of the solution.

1.4 Fixed Point (or Stationary Point or Equilibrium Point or Critical Point) [2]

A *fixed* point or an *equilibrium* point of a system of differential equations is a constant solution, that is, a solution X such that $X(t) = X(t_0)$ for all t.

If X is an equilibrium point, then we identify the equilibrium point with the vector $X(t_0)$. From the definition, it is clear that X is a fixed point of the system (1) if $X'(t) = 0$.

1.5 Classification of Fixed Points Depending Upon Their Stability [1]

Let X^* be a fixed point of a system $X' = F(X)$.

(i) We say that X^* is an attracting or stable fixed point if there is a $\delta > 0$ such that $\lim_{t\to\infty} X(t) = X^*$ whenever $\| X(0) - X^* \| < \delta$.

This definition implies that any trajectory that starts near X^* within a distance δ is guaranteed to converge to X^*eventually.

(ii) X^* is said to be Liapunov stable if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\| X(t) - X^* \| < \epsilon$ whenever $t \geq 0$ and $\| X(0) - X^* \| < \delta$.

Thus trajectories that start near X^* within δ remain within ϵ for all positive time. Liapunov stability requires that the nearby trajectories stay close for all the time.

(iii) The fixed point X^* is said to be asymptotically stable if it is both attracting and Liapunov stable.

II. THE DUFFING OSCILLATOR

In the field of nonlinear equations, van der Pol equation[5] is extensively studied. The equation is a mathematical modeling of the oscillating charge of the van der Pol oscillator. A strange dynamical behavior is observed in nonlinear oscillators with varying parameters. In this section, we will study the Duffing oscillator. Consider a periodically driven pendulum as shown in the figure 1.

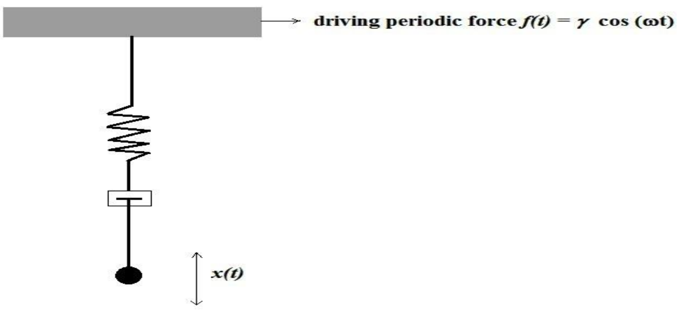


Figure 1.

Let $x(t)$ denote the displacement at time t from the rest position of the bob of the pendulum and let $\frac{dx}{dt} = x^{'}(t)$ denote the speed. Let $f(t) = \gamma \cos(\omega t)$ represent the driving periodic force, where γ is the driving amplitude and ω is the frequency of the driving force. Let q denote the damping coefficient and assume that the pendulum has a cubic restoring force. Then the Duffing equation representing the motion of the oscillator is given by

$$\frac{d^2x}{dt^2} + q\frac{dx}{dt} + (x^3 - x) = \gamma\cos(\omega t) \qquad (3)$$

A wide range of oscillators of this type are extensively studied so far and their behavior in terms of the nature of the solutions, their stability, chaotic nature and its control, etc. is examined. The authors Kulkarni P. R. and Borkar V. C.[8] have analyzed the oscillations in a damped driven pendulum and proved the chaotic nature of the pendulum oscillations. In this paper, we will study the solutions of the equation (3) by varying the damping amplitude γ while keeping the other parameters q and ω constants. For the sake of convenience we will choose $q = 0.3$ and $\omega = 1.25$. Despite of the equation (3) being two dimensional, it is not linear as it contains the periodic term $\cos(\omega t)$. Such a periodically forced non-autonomous differential equation can be represented by an autonomous differential equation by the introduction of a third variable $\theta = \omega t$. In this case, equation (3) can be expressed as a system of three first order differential equations given by $\frac{d\theta}{dt} = \omega, \frac{dx}{dt} = y$,

$\frac{dy}{dt} = -ky + x(1 - x^2) + \gamma\cos(\omega t)$.

The theory for autonomous and non-autonomous systems with reference to the nature of the solutions and their long term effect, the stability of fixed points, the nature of the trajectories and the phase portraits, etc. does not differ on a large scale. We will consider only the non-autonomous system.

Defining $x(t) = x_1(t), \frac{dx}{dt} = x_1{}' = x_2 = y$, equation (3) can be expressed as a system of differential equations

$$x_1{}' = x_2 \qquad (4)$$

$$x_2{}' = -0.3x_2 + x_1 - x_1^3 + \gamma\cos(1.25t) \qquad (5)$$

This system can be expressed in the form $X^{'}(t) = F(X,t)$, where

$$X^{'}(t) = \begin{bmatrix} f_1(x_1, x_2, t) \\ f_2(x_1, x_2, t) \end{bmatrix} = \begin{bmatrix} x_2 \\ (1 - x_1^2)x_1 - 0.3x_2 + \gamma\cos(1.25t) \end{bmatrix} \qquad (6)$$

The system (6) is a nonlinear nonautonomous system.

Taking $\gamma = 0$, the system of equations (4)-(5) can be expressed as

$$X^{'}(t) = F(X) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}, \qquad (7)$$

where $f_1(x_1, x_2) = x_2$ and $f_2(x_1, x_2) = (1 - x_1^2)x_1 - 0.3x_2$.

Solving the equation $X^{'}(t) = 0$ we get three equilibrium points $O = (0,0)$, $P = (1,0)$ and $Q = (-1,0)$. We will verify the nature of these equilibrium points in reference to their stability. The derivative $DF(X)$ of the function F at $X = (x_1, x_2)$ is given by

$$DF(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & -0.3 \end{bmatrix}$$

The linearized form of the system near the origin $O = (0,0)$ takes the form $X^{'} = AX$, where $A = DF(O) = \begin{bmatrix} 0 & 1 \\ 1 & -0.3 \end{bmatrix}$. The eigenvalues of the matrix A are -1.1611 and 0.8611. Since the eigenvalues are real with opposite signs, the equilibrium point $O = (0, 0)$ is a saddle point of the linearized system $X' = AX$. Similarly the matrices $DF(P) = DF(Q) = \begin{bmatrix} 0 & 1 \\ -2 & -0.3 \end{bmatrix}$ have eigenvalues $-0.15 \pm 1.40i$. As all the eigenvalues of both the matrices have negative real part, it follows that the equilibrium points $P = (1, 0)$ and $Q = (-1, 0)$ are both sinks for the linearized system $X' = AX$. As the fundamental theory for a linearized and a nonlinear system are qualitatively the same, by the Hartman-Grobman theorem, the origin $O = (0, 0)$ is a saddle point and the points P and Q are the sinks for the system (7).

The solutions for different initial conditions near the origin is as shown in Figure 2 and the phase plane portrait is as shown in the Figure 3. It can be observed that the orbits near the origin are moving away from the origin, in fact they are converging to the other two fixed points $P = (1, 0)$ and $Q = (-1, 0)$.

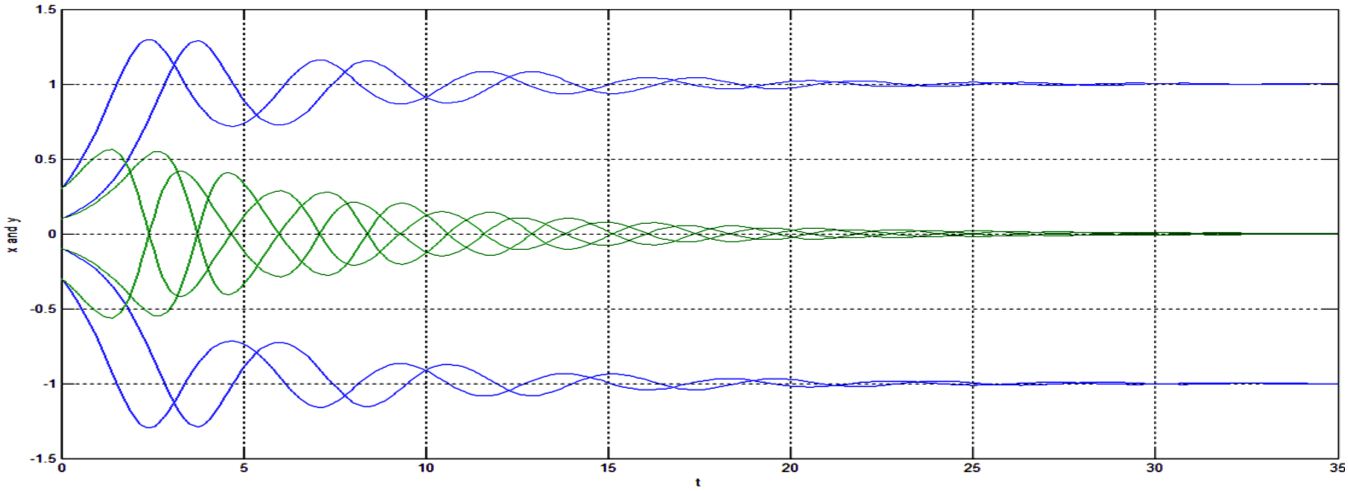


Figure 3. Trajectories starting at $(x, y) = (\pm0.1, \pm0.1)$ and $(x, y) = (\pm0.3, \pm0.3)$

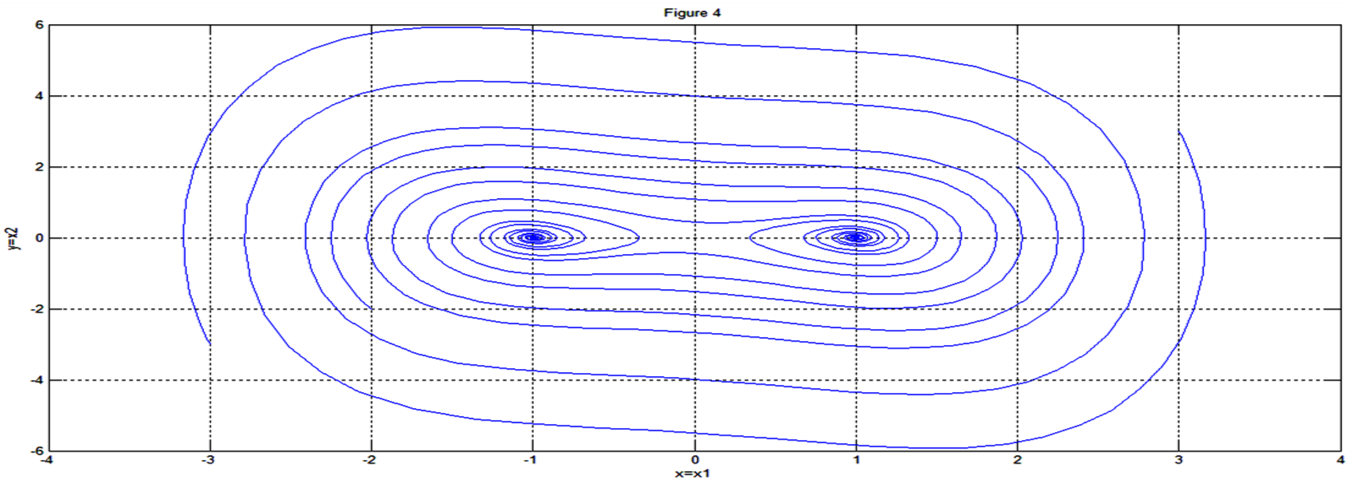


Figure 4. Phase plane portrait with initial conditions $(x, y) = (2,\ 2),\ (-2, -2),\ (3,\ 3),\ (-3, -3)$

In search of chaos, let us keep varying γ. We will now study the behavior of the system for $\gamma = 0.2$. In this case, solving the system of equations (4)-(5), after the initial transient is settled, it can be observed from Figure 4 and Figure 5 that the solution curves are harmonic with period equal to that of the driven force $i.e. \frac{2\pi}{\omega} \cong 5.026$. Figure 4 and Figure 5 are obtained by using different mathematical softwares.

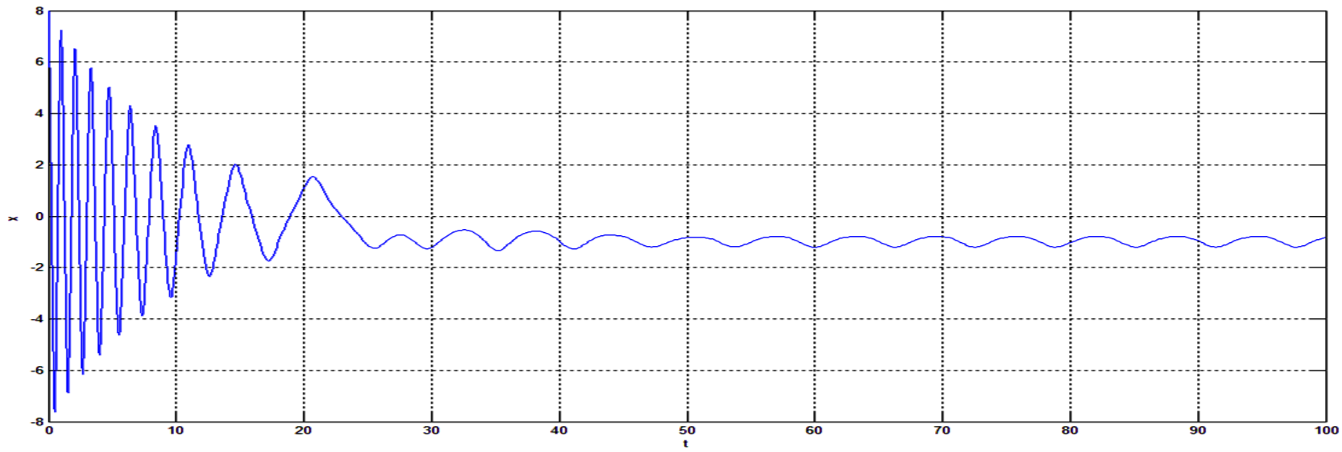


Figure 4: period-one harmonic solution for $\gamma = 0.2$

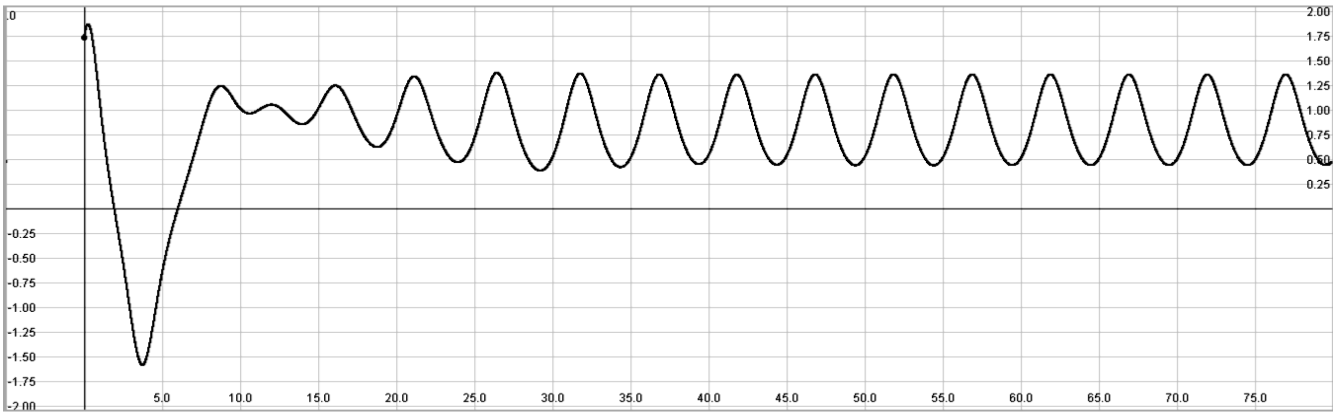


Figure 5: *t* v/s *x(t)*

For non-linear systems, sometimes exact solutions may not exist, and so we use numerical methods to obtain the solutions. It is quite difficult to obtain the trajectories by analytical methods, and though the trajectories are obtained by explicit formulas, they are too complicated to provide any kind of information about the solution itself. Hence qualitative behaviors of the trajectories obtained by numerical solution methods are often studied. To study the qualitative behavior of the solution, we obtain a phase portrait in which we plot the variable x_1 against the variable x_2 as the time t varies. The phase portrait and the vector field for $\gamma = 0.2, x_1 = 1.7, x_2 = 1.2$ is as shown in the Figure 6. The period-one harmonic solution can be verified by means of a closed curve in the phase portrait.

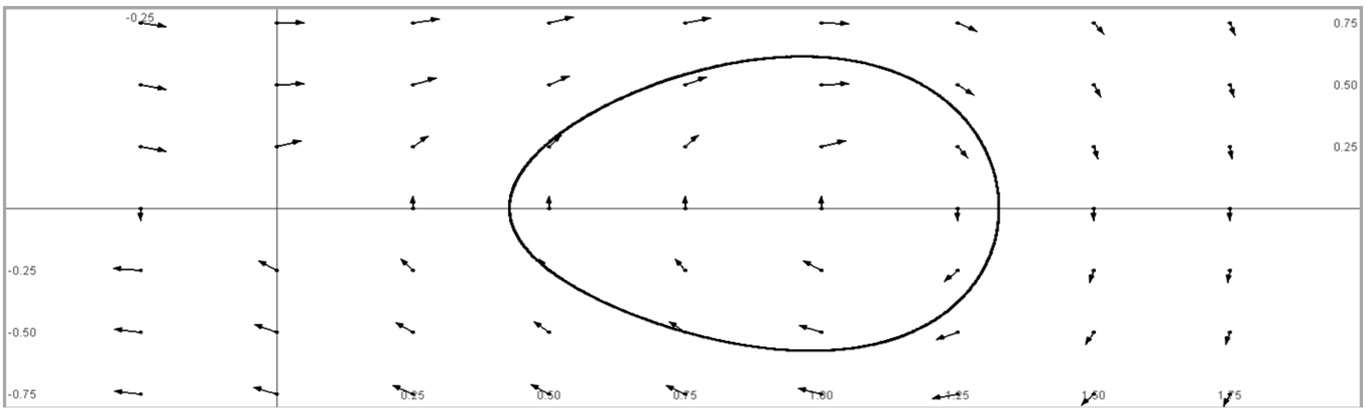


Figure 6: The phase portrait and the vector field for $\gamma = 0.2, x_1 = 1.7, x_2 = 1.2$

When we plot solution curves of some nonlinear system, the trajectories may cross each other and it becomes very difficult to draw any conclusions from them. Poincare sections of the phase portraits are often used in such situations. Poincare sections help us to observe the flow under consideration is a better way. The Poincare section in this case is as shown in the Figure 6a. A single point can be observed in the Poincare section.

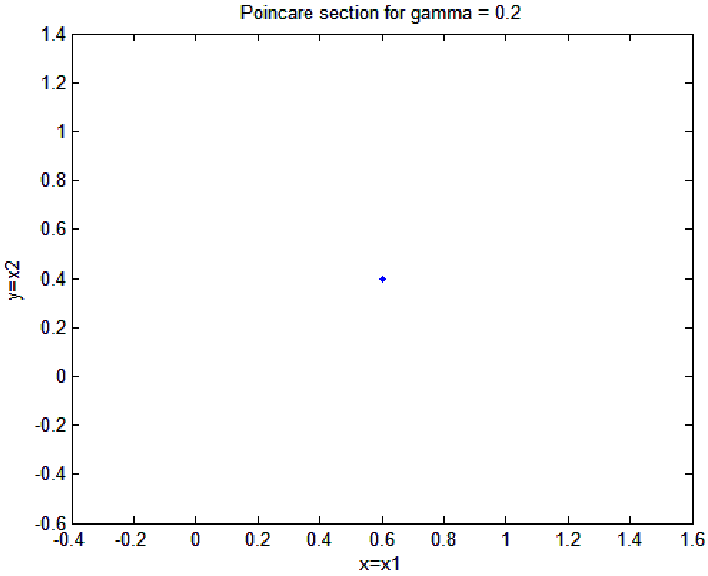


Figure 6a

For $\gamma = 0.3$, solutions harmonic with period equal twice the period of the driven force $i.e. 2.\frac{2\pi}{\omega} \cong 10.0531$ as can be observed from the Figure 7. A period two cycle can be observed in the phase portrait in this case as can be observed from the Figure 8. Note that the two trajectories crosses itself. The Figure 8a shows the Poincare section in which two points can be observed.

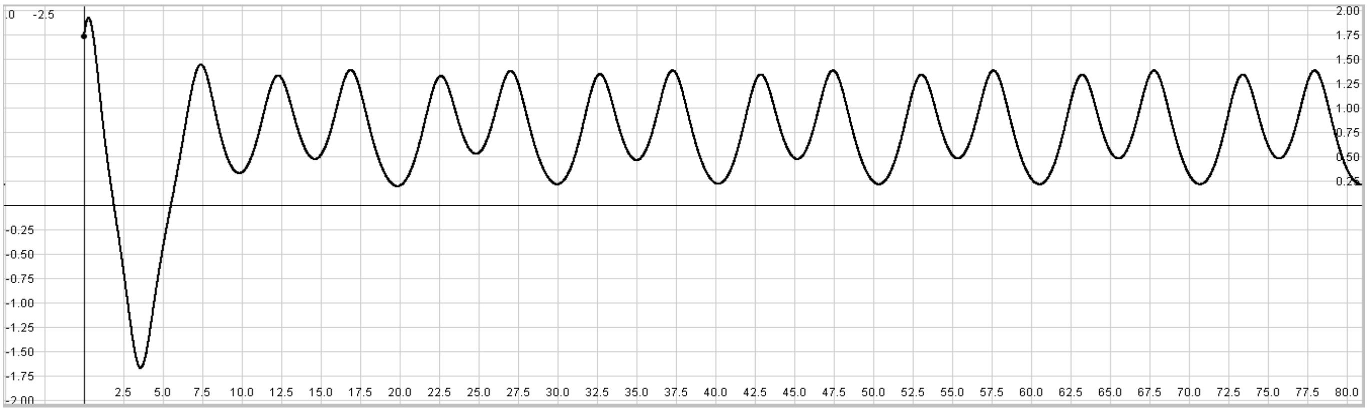


Figure 7: t v/s $x(t)$

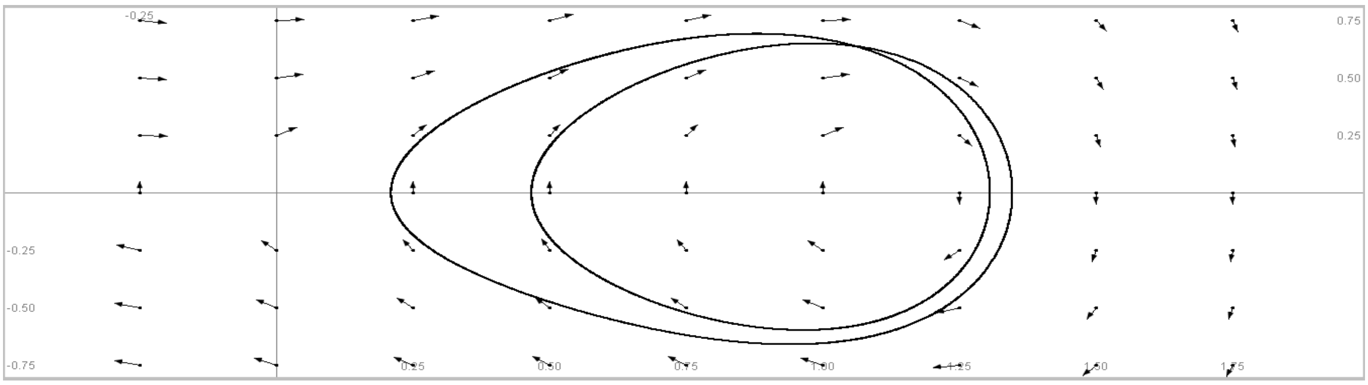


Figure 8: The phase portrait and the vector field for $\gamma = 0.3, x_1 = 1.7, x_2 = 1.2$

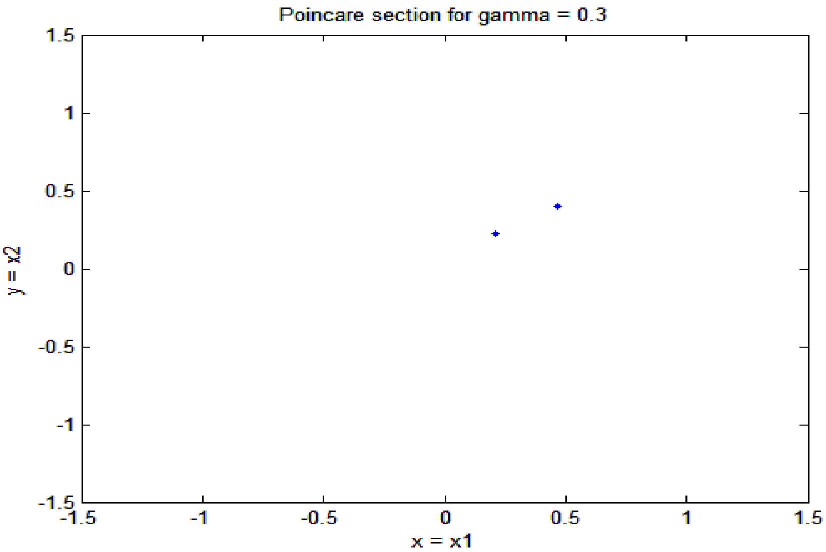


Figure 8a

Considering the value $\gamma = 0.31$, a period four cycle of period $4.\frac{2\pi}{\omega} \cong 20.106$ is observed as can be verified from the Figure 9. The phase portrait is shown by the Figure 10, where we can see a period four loop. Figure 11 shows the zoom in picture in this case. The Figure 11a shows the Poincare section in which four points can be observed.

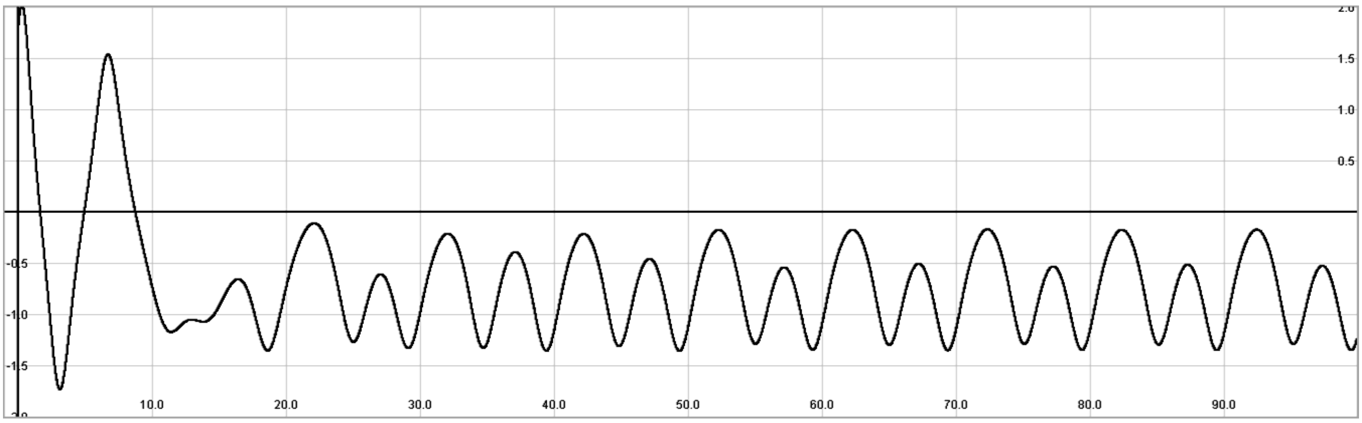


Figure 9: time t v/s x for $\gamma = 0.31, x_1 = 1.8, x_2 = 1.5$

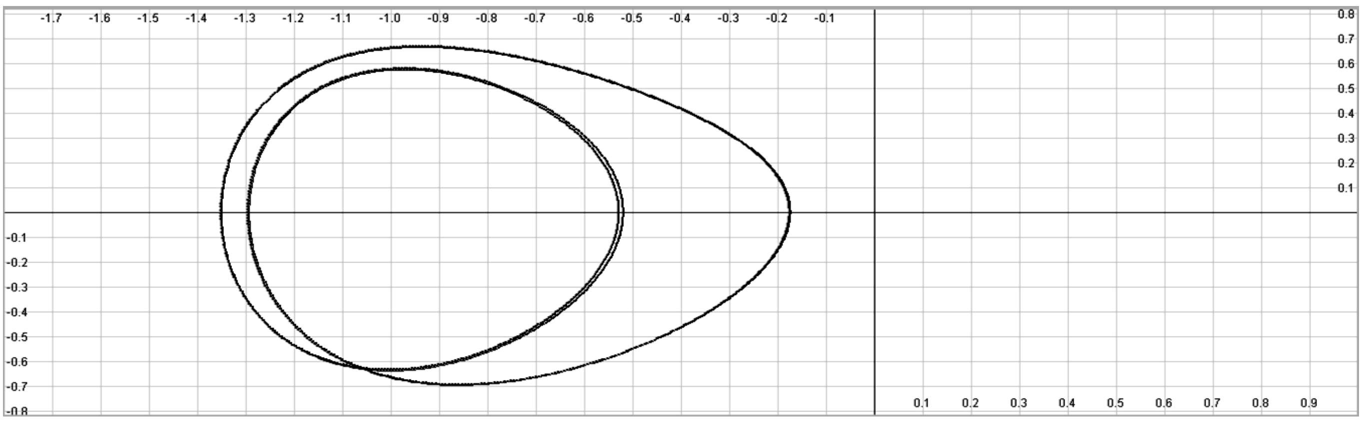


Figure 10: The phase portrait and the vector field for $\gamma = 0.31, x_1 = 1.8, x_2 = 1.5$

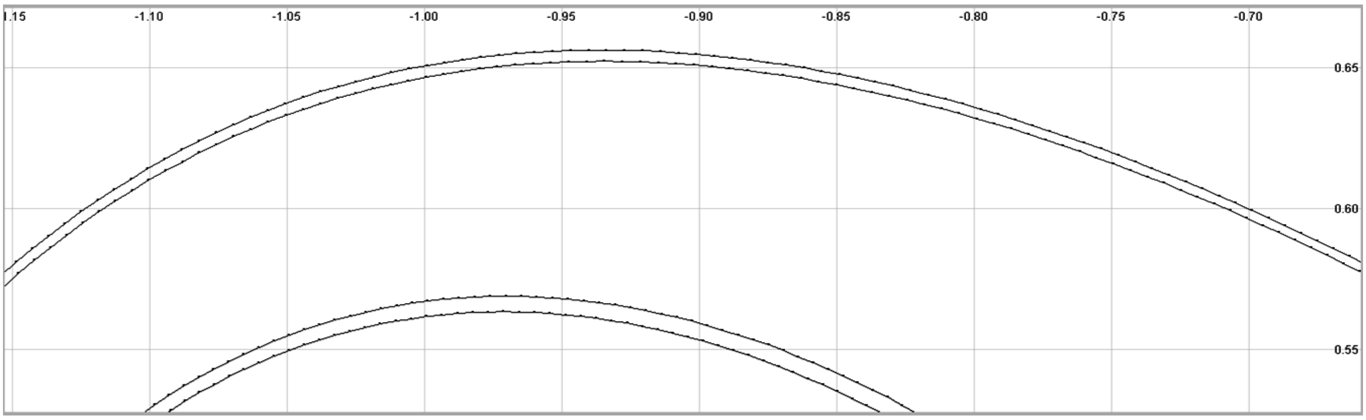
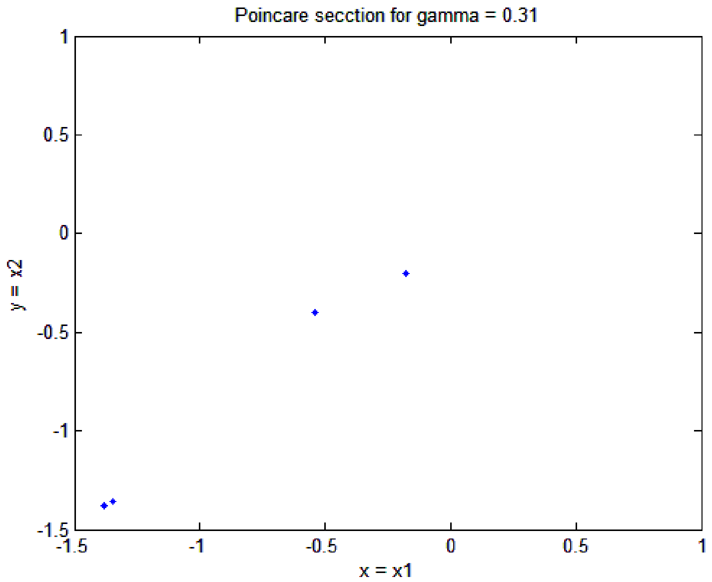


Figure 11: Zoom in on the phase portrait for $\gamma = 0.31, x_1 = 1.8, x_2 = 1.5$



III. RESULTS

The term *chaos*[6,7] is used when there is predictability in a system, but a kind of randomness or uncertainty for certain parameter ranges also. Chaotic behavior is quite observed in so many nonlinear systems representing a natural phenomenon. There are many definitions of chaos given by different authors including measure theoretic notions, topological concepts, etc. However, in accordance with the definition given by Devaney R. L. [3], the concepts involved in its definition are sensitive dependence on initial conditions, topological transitivity, and the denseness of the periodic orbits. One of the major characteristics of a chaotic system is the so called period doubling phenomenon[8] for certain range of the parameter. Note that for $\gamma = 0$, the solution curves are quite predictable. There are three equilibrium points and all the trajectories converge to only two equilibrium points $P = (1, 0)$ and $Q = (-1, 0)$ without crossings between them. For $\gamma = 0.2$, solution curves are harmonic with period equal to that of the driven force *i.e.*$\frac{2\pi}{\omega}$, $\gamma = 0.3$, solutions harmonic with period equal twice the period of the driven force *i.e.*$2.\frac{2\pi}{\omega}$. Thus there is a period doubling of the cycles. For $\gamma = 0.31$, there is a period four cycle, again a period doubling! As we go on increasing the values of the parameter γ, this period doubling phenomenon is not observed and the system enters in the chaotic regime. For $\gamma = 0.5$, the system becomes chaotic and a solution curve intersects itself many times. This can be observed from the phase portrait for as shown by Figure 12. However, this phase portrait is not the actual portrait as the system is nonautonomous. The actual phase portrait should be three dimensional depending on (x, y, t) co-ordinates and not only upon (x, y) co-ordinates. In fact, the Figure 12 is a projection of the actual phase portrait on the xy-plane.

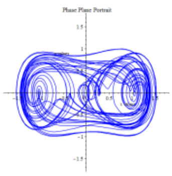


Figure 12: Phase portrait of the chaotic system

The Poincare section of the phase portrait is as shown in the following Figure 13. This section has fractal dimension which is a cross section of the strange attractor.

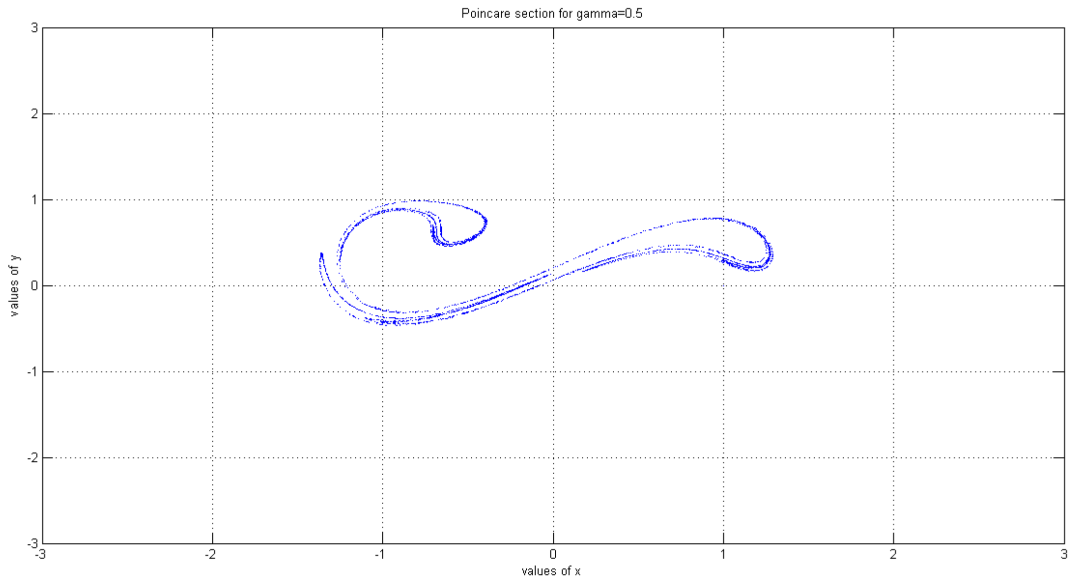


Figure 13: Poincare section of the strange attractor

The period doubling phenomenon can also be observed by means of the bifurcation diagram. By means of a bifurcation diagram, we can observe the values of the parameter γ at which the dynamical system bifurcates. This kind of diagram enables us to understand the behavior of the system at higher iterates at arbitrary initial conditions for all values of the parameter. In such a diagram, the values of the parameter γ are plotted on the horizontal axis and the higher iterates are plotted on the vertical axis. The bifurcation diagram of the system of equations (4)-(5) is as shown in the Figure 14.

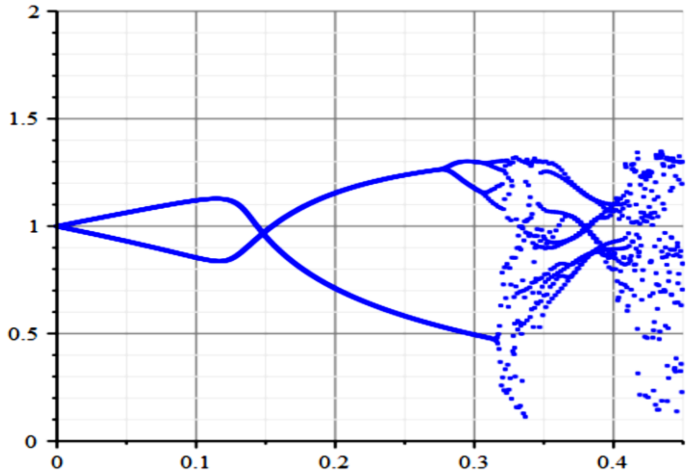


Figure 14: Bifurcation diagram

IV. CONCLUSIONS

From the bifurcation diagram, we can observe that there is a period-1 harmonic solution in the approximate range $0 < \gamma < 0.27$ and a period-2 harmonic solution in the approximate range $0.27 < \gamma < 0.32$. As γ increases further, an unpredictable behavior and then again, a periodic behavior is observed. It is very difficult to have predictions about the state of the system at a particular instant in case of such dynamical systems in chaotic region as there is a sensitive dependence on the initial conditions.

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