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Generalized Fractional Fourier-Wavelet Transform and its Applications

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ABSTRACT

The Fractional Fourier transform is a time frequency distribution and an extension of the classical Fourier transform. There are several known applications of the Fractional Fourier transform in areas of signal processing especially in signal restoration and noise removal. The Wavelet transform is useful for image processing to accurately analyze the abrupt changes in the image that is localize means in time and frequency. The aim of this paper is to present generalization of fractional Fourier-Wavelet transform.

Keyword: Fractional Fourier Transform, Wavelet Transform, Fractional Fourier-Wavelet Transform, Testing Function Space, Signal Processing

I. INTRODUCTION

In very simple terms, the fractional Fourier transform (FRFT) is a generalization of the ordinary Fourier transform [1]. Specifically, the FRFT implements the so-called order parameter α which act on the ordinary Fourier transform operator. In other words, the $\propto$-th order fractional Fourier transform represents the $\propto$-th power of the ordinary Fourier transform operator. When $\propto = \pi/2$, we obtain the Fourier transform, while for $\alpha = 0$ 0, we obtain the signal itself. Any intermediate value of a $\propto (0 < \propto < \pi/2)$ produces a signal representation that can be considered as a rotated time–frequency representation of the signal [2][3].

The fractional Fourier transforms was introduced by Namias as a Fourier transform of fractional order. The most important aspect of fractional Fourier transform is its applicability in time varying signals for which the Fourier transforms fails to work. Fractional Fourier transform was first introduced as a way to solve certain classes of ordinary and partial differential equation arising in quantum mechanics. Fractional Fourier transform has established itself as a powerful tool for the analysis of time varying signals, especially in optics. Fractional Fourier transform has found applications in areas of signal processing such as repeated filtering, fractional convolution, beam forming, optimal filter, convolution, filtering and wavelet transform, time frequency representation [4][5].

The wavelet transform theory is based on analysis of signal using varying scales in the time domain and frequency. Formalization was carried out in the 1980s, based on the generalization of familiar concepts. The

wavelet term was introduced by French geophysicist Jean Morlet. The seismic data analyzed by Morlet exhibit frequency component that changed rapidly over time, for which the Fourier Transform (FT) is not appropriate as an analysis tool. Thus, with the help of theoretical physicist Croatian Alex Grossmann, Morlet introduced a new transform which allows the location of high-frequency events with a better temporal resolution [6].
The wavelet transform (WT) has been shown to be an appropriate tool for time-frequency analysis. Wavelet transform has been applied in many fields of signal processing, including speech, image, communications, radar. The wavelet transform has been shown to be a successful tool for dealing with transient signals, data compression, bandwidth reduction, and time-dependent frequency analysis of short transient signals, [4] optical correlators, [5], [6] sound analysis, [7] representation of the human retina, and representation of fractal aggregates[8].Mathematically, the wavelet operation is equivalent to performing a Fourier transform of the input function, multiplying it by a differently scaled Fourier transforms of the wavelet mother function, and eventually performing an inverse Fourier transform[9].

II. THE CONVENTIONAL FRACTIONAL FOURIER-WAVELET TRANSFORM

The conventional Fractional Fourier-Wavelet transform is defined as

$$FrFT\{f(x,t)\} = F_\alpha(p,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)K_\alpha(x,p,t,a,b)dxdt$$

where $K_\alpha(x,p,t,a,b) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}\frac{1}{\sqrt{|a|}}e^{i\pi\left(\frac{t-b}{a}\right)^2}$ -----2.1

where $0 < \alpha < \frac{\pi}{2}$.

$K_\alpha(x,p,t,a,b)$ belongs to the testing function space and $f(x,\mathrm{t})$ lies in its dual space.

III. THE TESTING FUNCTION SPACE E($\mathbf{R^n}$)

An infinitely differentiable complex valued smooth function $\emptyset(x,t)$ on R^n belongs to $E(R^n)$, if for each compact set I $\subset S_{c,d}$,

$$S_{c,d} = \{x,t: x,t \in R^n, |x| \le c,\ |t| \le d,\ c > 0,\ d > 0\}$$

$$\gamma_{E,l,n}(\emptyset) = \sup_{\substack{x\in I\\ t\in I}}\left|D_x^l D_t^n \emptyset(x,t)\right| < \infty \qquad \text{-----3.1}$$

Thus, $E(R^n)$ will denotes the space of all $\emptyset \in E(R^n)$ with support contained in $S_{c,d}$. Moreover, we say that f is a fractional Fourier-Wavelet Transformable if it is a member of E^*,the dual space of E.

IV. DISTRIBUTIONAL FRACTIONAL FOURIER-WAVELET TRANSFORM

The distributional Fractional Fourier –Wavelet transform of $f(x,t) \in E(R^n)$ defined by

$$FrFWT\{f(x,t)\} = F_\alpha(p,b) = \langle f(x,t), K_\alpha(x,p,t,a,b)\rangle$$

where $K_\alpha(x,p,t,a,b) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} \times \frac{1}{\sqrt{|a|}}e^{i\pi\left(\frac{t-b}{a}\right)^2}$

$$K_\alpha(x,p,t,a,b) = A_\alpha\, e^{iB_\alpha[(x^2+p^2)cos\alpha-2xp]} \times C_\alpha\, e^{i\pi\left(\frac{t-b}{a}\right)^2} \qquad \text{-----4.1}$$

$A_\alpha = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $B_\alpha = \frac{1}{2sin\alpha}$, $C_\alpha = \frac{1}{\sqrt{|a|}}$

V. ANALYTICITY THEOREM

Statement: Let $f(x,t) \in E^*(R^n)$ and let its fractional Fourier-Wavelet transform be defined as

$$FrFWT\{f(x,p,t,a,b)\} = F_\alpha(p,b) = \langle f(x,t), K_\alpha(x,p,t,a,b)\rangle$$

Then $F_\alpha(p,b)$ is analytic on C^n if the $suppf \subset S_{c,d}$ such that

$$S_{c,d} = \{x,t{:}\, x,t \in R^n, |x| \le c,\ |t| \le d,\ c > 0,\ d > 0\}$$

Moreover $FrFWT_\alpha(p,a,b)$ is differentiable and $D_x^l D_t^n(F_\alpha(p,b)) = \left\langle f(x,t), D_x^l D_t^n K_\alpha(x,p,t,a,b)\right\rangle$

where $K_\alpha(x,p,t,a,b) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} \times \frac{1}{\sqrt{|a|}} e^{i\pi\left(\frac{t-b}{a}\right)}$

Proof: Let $p = (p_1, p_2, p_3, ---p_i, ---p_n) \in C^n$ and $b = (b_1, b_2, b_3, ---b_i, ---b_n) \in C^n$ be arbitrary but fixed point and 'a' is fixed point.

We first prove that,

$$\frac{\partial}{\partial p_i}\{F_\alpha(p,b)\} = \langle f(x,t), \frac{\partial}{\partial p_i} K_\alpha(x,p,t,a,b)\rangle$$

For fixed, $p_i \neq 0$ choose two concentric circles C' and C'' with centre at p_i with radii r and r_1 respectively such that $0 < r < r_1 < |p_i|$,

Let Δp_i be a complete increment satisfying $0 < |\Delta p_i| < r$

Consider

$$\frac{F_\alpha(p_i + \Delta p_i, b) - F_\alpha(p_i, b)}{\Delta p_i} - \langle f(x,t), \frac{\partial}{\partial p_i} K_\alpha(x,p,t,a,b)\rangle = \langle f(x,t), \psi_{\Delta p_i}(x,t)\rangle \qquad \text{-----5.1}$$

$$\langle f(x,t), \psi_{\Delta p_i}(x,t)\rangle = \frac{1}{\Delta p_i}\{\langle f(x,t), K_\alpha(x, p_i + \Delta p_i, t, a, b)\rangle - \langle f(x,t), K_\alpha(x, p_i, t, a, b)\rangle\} - \langle f(x,t), \frac{\partial}{\partial p_i} K_\alpha(x,p,t,a,b)\rangle$$

$$\langle f(x,t), \psi_{\Delta p_i}(x,t)\rangle = \langle f(x,t), \frac{1}{\Delta p_i}\{K_\alpha(x, p_i + \Delta p_i, t, a, b) - K_\alpha(x, p_i, t, a, b)\} - \frac{\partial}{\partial p_i} K_\alpha(x,p,t,a,b)\rangle \langle f(x,t), \psi_{\Delta p_i}(x,t)\rangle$$

where $\psi_{\Delta p_i}(x,t) = \frac{1}{\Delta p_i}\{K_\alpha(x, p_i + \Delta p_i, t, a, b) - K_\alpha(x, p_i, t, a, b)\} - \frac{\partial}{\partial p_i} K_\alpha(x,p,t,a,b)$

for any fixed $x, t \in R^n$ and fixed integer $l = (l_1, l_2, l_3 --- l_n)$

$$D_x^l\{K_\alpha(x,p,t,a,b)\} = D_x^l\left\{\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} \times \frac{1}{\sqrt{|a|}} e^{i\pi\left(\frac{t-b}{a}\right)^2}\right\}$$

$$= D_x^l\left\{\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} \times B(t)\right\}$$

where $B(t) = \frac{1}{\sqrt{|a|}} e^{i\pi\left(\frac{t-b}{a}\right)^2}$

$$D_x^l\{K_\alpha(x,p,t,a,b)\} = A_\alpha\left[\sum_{k=0}^{\lfloor l/2\rfloor} \frac{l!}{k!\,(l-2k)!}(2iB_\alpha)^{l-k}\left(\frac{cos\alpha}{2}\right)^k (xcos\alpha - p)^{l-2k}\right]$$

$$e^{iB_\alpha[(x^2+p^2)cos\alpha-2xp]}B(t)$$

where $A_\alpha = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $B_\alpha = \frac{1}{2sin\alpha}$

$$D_x^l\{K_\alpha(x,p,t,a,b)\} = \left[\sum_{k=0}^{\lfloor l/2\rfloor}\frac{l!}{k!\,(l-2k)!}(2iB_\alpha)^{l-k}\left(\frac{cos\alpha}{2}\right)^k(xcos\alpha-p)^{l-2k}\right]$$
$$\sqrt{\frac{1-icot\alpha}{2\pi}}\,e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}\frac{1}{\sqrt{|a|}}e^{i\pi\left(\frac{t-b}{a}\right)^2}$$
$$= \left[\sum_{k=0}^{\lfloor l/2\rfloor}\frac{l!}{k!\,(l-2k)!}(2iB_\alpha)^{l-k}\left(\frac{cos\alpha}{2}\right)^k(xcos\alpha-p)^{l-2k}\right]K_\alpha(x,p,t,a,b)$$

------5.2

Since for any fixed $x \in R^n$ and fixed l and $0 < \alpha \le \pi/2$
$D_x^l\{K_\alpha(x,p,t,a,b)\}$ is analytic inside and on C',
We have by Cauchy integral formula,

$$D_x^l\psi_{\Delta p_i}(x,t) = \frac{1}{2\pi i}D_x^l\int_{C'}K_\alpha(x,\tilde{p},t,a,,b)\left[\frac{1}{\Delta p_i}\left(\frac{1}{z-p_i-\Delta p_i}-\frac{1}{z-p_i}\right)-\left(\frac{1}{z-p_i}\right)^2\right]dz$$

where $\tilde{p} = (p_1,p_2,p_3,---p_{i+1},z,p_{i+1},---p_n)$

$$D_x^l\psi_{\Delta p_i}(x,t) = \frac{1}{2\pi i}D_x^l\int_{C'}K_\alpha(x,\tilde{p},t,a,b)\left[\left(\frac{1}{(z-p_i-\Delta p_i)(z-p_i)}\right)-\left(\frac{1}{z-p_i}\right)^2\right]dz$$

$$D_x^l\psi_{\Delta p_i}(x,t) = \frac{\Delta p_i}{2\pi i}D_x^l\int_{C'}\left[\left(\frac{D_x^lK_\alpha(x,\tilde{p},t,a,b)}{(z-p_i-\Delta p_i)(z-p_i)^2}\right)\right]dz$$

$$D_x^l\psi_{\Delta p_i}(x,t) = \frac{\Delta p_i}{2\pi i}D_x^l\int_{C'}\left[\left(\frac{Q(x,\tilde{p},t,a,b)}{(z-p_i-\Delta p_i)(z-p_i)^2}\right)\right]dz$$

----5.3

But for all $z \in C'$ and x is restricted to a compact subset of $R^n, 0 < \alpha \le \pi$,
$Q(x,\tilde{p},t,a,b) = D_x^lK_\alpha(x,\tilde{p},t,a,b)$ as in 5.3 bounded by a constant L.
Moreover, $|z-p_i-\Delta p_i| > r_1 - r > 0$ and $|z-p_i| = r_1$ therefore we have

$$\left|D_x^l\psi_{\Delta p_i}(x,t)\right| = \left|\frac{\Delta p_i}{2\pi i}D_x^l\int_{C'}\left[\left(\frac{Q(x,\tilde{p},t,a,b)}{(z-p_i-\Delta p_i)(z-p_i)^2}\right)\right]dz\right|$$
$$\le \frac{|\Delta p_i|}{2\pi}\int_{C'}\frac{L}{(r_1-r){r_1}^2}|dz|$$
$$\le \frac{|\Delta p_i|}{2\pi}\frac{L}{(r_1-r){r_1}^2}2\pi r_1$$
$$\left|D_x^l\psi_{\Delta p_i}(x,t)\right| \le \frac{|\Delta p_i|L}{(r_1-r)r_1}$$

Similarly $\left|D_t^n\psi_{\Delta b_i}(x,t)\right| \le \frac{|\Delta b_i|M}{(r_1-r)r_1}$

where $P(x,p,t,a,\tilde{b}) = D_t^n K_\alpha(x,p,t,a,\tilde{b})$ is bounded by a Constant M.

Thus as $|\Delta p_i| \to 0$ and $|\Delta b_i| \to 0$. $\left|D_x^l \psi_{\Delta p_i}(x,t)\right|$ and $\left|D_t^n \psi_{\Delta b_i}(x,t)\right|$ tends to zero uniformly on the compact subset of R^n, therefore it follows that $\psi_{\Delta p_i}(x,t)$ and $\psi_{\Delta b_i}(x,t)$ converges in $E(R^n)$ to zero. Since $f(x,t) \in E^*$, we conclude that equation (A) also tends zero. Therefore $F_\alpha(p,b)$ is Differentiable with respect to p_i and b_i.But this is true for all $i = 1,2,3,---n$, hence $F_\alpha(p,b)$ is analytic on C^n and $D_x^l D_t^n\ (\ F_\alpha(p,b)) = \langle f(x,t), D_x^l D_t^n K_\alpha(x,p,t,a,b)) \rangle$.

VI. CONCLUSION

In this paper Fractional Fourier –Wavelet Transform is introduced, Analyticity theorem for Fractional Fourier-Wavelet transform is proved.

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