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Analytical Structure of Mellin-Wavelet Transform

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ABSTRACT

Wavelets are mathematical tool which can be used to extract information from many different kinds of data including audio signals and images. The Wavelet transform decomposed the signal with finite energy in the spatial domain into a set of functions. The Wavelet transform has been shown to be a successful tool for dealing with transient signals, data compression, sound analysis, representation of the human retina. Mellin transform, a kind of nonlinear transformation, is widely used for its scale invariance property. The main objective of this paper is the generalization of analytical structure of Mellin-Wavelet transform.

Keywords:- Mellin transform, Wavelet transform, Testing function space, Mellin-Wavelet transform, signal processing.

I. INTRODUCTION

The word "Wavelet" has been introduced by Morlet and Grossmann [1] in the early 1980s. A Wavelet is a wave-like oscillation that is localized in time. The main objective of Wavelet Transform is to define the powerful wavelet basis functions and find efficient methods for their computation. The wavelets, which are based on scale invariance and self-similarity-fractal patterns, are therefore the most suitable technique employed to study the biomedical and other texture images. Being a versatile tool especially for the analysis of quasi-chaotic signals, noisy images, wavelets have got applications in all branches of medicine, biology, computer tomology, analysis of ECG, brain wave studies. In biological systems, introducing stochastic 'noise' has been found helpful in improving the signal strength of the internal feedback loops for balance and other vestibular communication. It has been found helpful to diabetic and stroke patients with balance control.

The Mellin transform is an integral transform named after the finnish mathematician Hjalmar Mellin (1854-1933). Mellin transform is basic tool for analyzing the behaviour of many in mathematics and mathematical physics. Mellin transform is implemented as a fast Mellin transform [2]. Mellin transform is widely used for its scale invariance property [3-5]. Mellin transform has many applications such as navigation, radar system, in finding the stress distribution in an infinite wedge, also in digital audio effects [6]. Karen Kohl and Flavia Stan introduced an algorithmic approach to the Mellin transform method by applying Wegschaider's algorithm in

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his research work [7]. Mellin transform offers human a new way to figure problem out [8-11]. It was R. H. Mellin initially gave a systematic formulation of transform and its inverse. In this paper we introduced generalized Mellin-Wavelet transform and proved analyticity theorem for Mellin-Wavelet transform.

II. THE CONTINUOUS WAVELET TRANSFORM

The Continuous Wavelet transform of a decomposition function $f(x)$ with respect to a basic wavelet $\Psi(x)$ given by the convolution of a function with a scaled and translated version of $\Psi(x)$ is defined as

$$W_\Psi(a,b)[f] = \frac{1}{|a|^{1/2}}\int f(x)\Psi^*\left(\frac{x-b}{a}\right)dx = \langle f(x), \frac{1}{\sqrt{|a|}}\Psi\left(\frac{x-b}{a}\right)\rangle \qquad (I)$$

where $\langle\cdot,\cdot\rangle$ is the inner product.

The function f and Ψ are square integrable function and Ψ satisfies the admissibility condition

$$C_\Psi = \int \frac{|\hat{\Psi}(w)|^2}{|w|}\, dw < \infty$$

Subscript '$*$' denotes the complex conjugation, 'a' is the scale parameter $a > 0$; 'b' is the translation parameter. The term $\frac{1}{\sqrt{a}}$ is the energy conservative term that keeps the energy of the scaled mother wavelet equal to energy of the original wavelet [12].

The classical wavelet transform of a function f with respect to a given admissible mother wavelet is $\Psi(x) = exp(i\pi x^2)$ defined as wavelet domain coefficient at scalar parameter $a = tan\alpha^{\frac{1}{2}}$.

$$W_\Psi(a,b)[f] = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} f(x)\, e^{i\pi\left(\frac{x-b}{a}\right)^2} dx = \frac{1}{|a|^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(x)\, e^{i\pi\frac{(x-b)^2}{tan\alpha}} dx$$

where Ψ satisfies the admissibility condition

$$C_\Psi = \int \frac{|\hat{\Psi}(w)|^2}{|w|}\, dw < \infty$$

where C_Ψ is the admissibility constant.

III. MELLIN –WAVELET TRANSFORM

The Conventional Mellin –Wavelet transform is defined as

$$MW_\Psi\{f(t,x)\} = MW_\Psi(p,a,b) = \int_0^\infty \int_{-\infty}^\infty f(t,x)\, K(t,x,p,a,b)\, dt\, dx$$

where $K(t,x,p,a,b) = \frac{1}{|a|^{\frac{1}{2}}}\, t^{p-1}\, e^{i\pi\left(\frac{x-b}{a}\right)^2}$

IV. TESTING FUNCTION SPACE $MW_{a,b,\Psi,p}$

An infinitely differentiable complex valued smooth function $\phi(t,x,p,a,b)$ define over $-\infty < x < \infty, 0 < t < \infty$ with the parameter p,a,b is said to belong to $MW_{a,b,\Psi,p}$ for each $m,n \in \mathbb{R}^2$

$$\gamma_{l,k,p}\phi(t,x) = \operatorname*{Sup}_I \left|\xi_{m,n}(t)t^{q+1}D_t^q D_x^k \phi(t,x)\right| < \infty$$

where $q,k = 0,1,2,3,\ldots\ldots$

$$\xi_{m,n} = \begin{cases} t^{-m} & ,0 < t \le 1 \\ t^{-n} & ,1 < t < \infty \end{cases}$$

Now we prove the kernel of Mellin Wavelet transform belongs to the $W_{a,b,\alpha,p}$.

V. DISTRIBUTIONAL GENERALIZED MELLIN-WAVELET TRANSFORM

For $f(t,x) \in MW^*_{a,b,\Psi,p}$

where $\mathrm{MW}^*_{\mathrm{a,b,\Psi,p}}$ is the dual space of $MW_{a,b,\Psi,p}$ and $m < Re\, p < n$, $b \in R, a \neq 0$;

the distributional Mellin-Wavelet transform is defined as

$$\mathrm{MW}_\Psi\{f(t,x)\} = \mathrm{MW}_\Psi(p,a,\ b) = \langle f(t,x),\ K(t,x,p,a,b)\rangle \quad (1)$$

$$\text{where } K(t,x,p,a,b) = \frac{1}{|a|^{\frac{1}{2}}}\ t^{p-1}\ e^{i\pi\left(\frac{x-b}{a}\right)^2} \quad (2)$$

R. H. S. of equation(1) has a sense as an application of $f(t,x) \in MW^*_{a,b,\Psi,p}$ to $K(t,x,p,a,b) \in \mathrm{MW}_{\mathrm{a,b,\Psi,p}}$.

VI. ANALYTICITY THEOREM

Statement If $F(p,b) = \langle f(t,x), K(t,x,p,a,b)\rangle$ that is $F(p,a) = \langle f(t,x), \frac{1}{|a|^{\frac{1}{2}}}\ t^{p-1}\ e^{i\pi\left(\frac{x-b}{a}\right)^2}\rangle$ Then $F(p,b)$ is analytic for some fixed $p > 0, b > 0, a > 0$. and $\frac{\partial}{\partial p}\frac{\partial}{\partial b}F(p,b) = \langle f(t,x), \frac{\partial}{\partial p}\frac{\partial}{\partial b}K(t,x,p,a,b)\rangle$ where $K(t,x,p,a,b) = \frac{1}{|a|^{\frac{1}{2}}}\ t^{p-1}\ e^{i\pi\left(\frac{x-b}{a}\right)^2}$

Proof Let p and b be an arbitrary but fixed. Choose the real positive number a_1 , $b_{1,}$ *and* r such that $\sigma_1 < a_1 < p - r < p + r < b_1 < \sigma_2$.

Let Δp be a complex increment such that $0 < \Delta p < r$.

For $\Delta p \neq o$, we write

$$\frac{F(p+\Delta p,b)-F(a,b)}{\Delta p} - \langle f(t,x), \frac{\partial}{\partial p}\frac{1}{|a|^{\frac{1}{2}}}\ t^{p-1}\ e^{i\pi\left(\frac{x-b}{a}\right)^2}\rangle = \langle f(t,x), \frac{1}{|a|^{\frac{1}{2}}}\frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}}{\Delta p}\left[t^{(p+\Delta p)-1} - t^{p-1}\right] - \frac{\partial}{\partial p}\frac{t^{p-1}}{|a|^{\frac{1}{2}}}e^{i\pi\left(\frac{x-b}{a}\right)^2}\rangle = \langle f(t,x), \Psi_{\Delta p}(t,x)\rangle$$

$$\text{Where } \Psi_{\Delta p}(t,x) = \frac{1}{|a|^{\frac{1}{2}}}\frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}}{\Delta p}\left[t^{(p+\Delta p)-1} - t^{p-1}\right] - \frac{\partial}{\partial p}\frac{t^{p-1}}{|a|^{\frac{1}{2}}}e^{i\pi\left(\frac{x-b}{a}\right)^2}$$

To prove:

$\Psi_{\Delta p}(t,x) \in MW_{a,b,\Psi,p}$, we shall show that as $|\Delta p| \to 0$, $\Psi_{\Delta p}(t,x)$ converges in $MW_{a,b,\Psi,p}$ to zero.

To proceed, let C denotes the circle with centre at p and radius r_1 ,

Where $0 < r < r_1 < min\left(p - a_{1,}\ b_1 - p\right)$. We may interchange differentiation on p with differentiation on t.

$$(-D_t)^q\Psi_{\Delta p}(t,x)$$

$$= (-D_t)^q\left\{\frac{1}{|a|^{\frac{1}{2}}}\frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}}{\Delta p}\left[t^{(p+\Delta p)-1} - t^{p-1}\right] - \frac{\partial}{\partial p}\frac{t^{p-1}}{|a|^{\frac{1}{2}}}e^{i\pi\left(\frac{x-b}{a}\right)^2} - \frac{\partial}{\partial t}\frac{1}{|a|^{\frac{1}{2}}}\ t^{p-1}e^{i\pi\left(\frac{x-b}{a}\right)^2}\right\}$$

$$= \frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}}{|a|^{\frac{1}{2}}\,\Delta p}\left\{\mathrm{P}(p+\Delta p)\ t^{p+\Delta p-q-1} - P(p)t^{p-q-1}\right\}$$

Where $P(p+\Delta p)$ is polynomial in $p+\Delta p$ and $P(p)$ is polynomial in p.

Now applying Cauchy's integral formula.

$$(-D_t)^q\Psi_{\Delta p}(t,x)$$

$$= \frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}}{|a|^{\frac{1}{2}}\,\Delta p}\left\{\frac{1}{2\pi i}\int_C \frac{P(z)t^{z-q-1}}{(z-p-\Delta p)}\,dz - \frac{1}{2\pi i}\int_C \frac{P(z)t^{z-q-1}}{(z-p)}\,dz - \frac{1}{2\pi i}\int_C \frac{P(z)t^{z-q-1}}{(z-p)^2}\,dz\right\}$$

$$= \frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}}{2\pi i|a|^{\frac{1}{2}}} \int_c \left[\frac{1}{(z-p-\Delta p)(z-p)} - \frac{1}{(z-p)^2}\right] P(z)t^{z-q-1}dz$$

$$= \frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}\Delta p}{2\pi i|a|^{\frac{1}{2}}} \int_c \frac{1}{(z-p-\Delta p)(z-p)^2}\, P(z)t^{z-q-1}dz$$

$D_t^q D_x^l \Psi_{\Delta p,\Delta v}(t,x)$

$$= \frac{e^{i\pi\left(\frac{x-b}{a}\right)^2}(i\pi)^l \Delta p \mathrm{V(v)}}{2\pi i|a|^{\frac{1}{2}}} \int_c \frac{P(z)t^{z-q-1}\, dz}{(z-p-\Delta p)(z-p)^2}$$

Now for all $z \in C, -\infty < x < \infty, 0 < t < \infty$

$$\sup_{I}\left|\xi_{m,n}(t)t^{q+1}\, D_t^q D_x^l \Psi_{\Delta p}(t,x)\right| \le \mathrm{K}$$

where K is a constant independent of z and t.

Moreover $|z-p-\Delta p| > r_1 > r > 0$ and $|z-p| = r_1$

$\mathrm{C}_1 = \max\{|P(z)t^z| : z \in C\}$

Consequently

$$\sup_{I}\left|\xi_{m,n}(t)t^{q+1}(i\pi)^l e^{i\pi\left(\frac{x-b}{a}\right)^2}\Delta p \mathrm{V(v)} \int_c \frac{P(z)\, t^{z-q-1}\, dz}{(z-p-\Delta p)(z-p)^2}\right|$$

$$\le \sup_{I}\left|\xi_{m,n}(t)t^{q+1-q-1}(i\pi)^l e^{i\pi\left(\frac{x-b}{a}\right)^2}\mathrm{V(v)}\right| |\Delta p| \int_c \frac{|P(z)t^z|}{|z-p-\Delta p||z-p|^2}|dz|$$

$$\le \mathrm{K}\frac{|\Delta p|}{2\pi}\int_c \frac{\mathrm{C}_1}{(r_1-r){r_1}^2}|dz|$$

$$\le \frac{|\Delta p|}{2\pi}\frac{\mathrm{C}_2}{(r_1-r){r_1}^2}\int_c |dz|$$

$$\le \frac{|\Delta p|}{2\pi}\frac{\mathrm{C}_2}{(r_1-r){r_1}^2} 2\pi r_1 \qquad \text{where } \mathrm{c}_2 = \mathrm{kc}_1$$

$$\le \frac{|\Delta p|\, \mathrm{C}_2}{(r_1-r)r_1}$$

The right hand side is independent of t and converges to zero as $|\Delta p| \to 0$

This shows that $\Psi_{\Delta p,\Delta v}(t,x)$ converges to zero in $\mathrm{MW}_{a,b,\Psi,p}$ as $|\Delta p| \to 0$

VII. CONCLUSION

In the present work distributional generalization of Mellin-Wavelet transform is presented. Analyticity theorem for Mellin-Wavelet transform is proved.

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