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New Approach / Definition of Fractional Derivative i.e Generalization of Comfortable Fractional Derivative

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ABSTRACT

In this paper we define new definition of fractional derivative i.e Generalize comformable fractional derivative and verify its validity for Linearity property, product rule, Quotient rule, and verify derivative of some Standard function. The definition satisfies the previous results for ordinary derivative and derivative of some standard function.
Keywords: Fractional Derivative, New Approach of Fractional Derivative.

I. INTRODUCTION

Fractional derivative is as old as calculus. L'Hospital in 1695 asked what does it mean if $\frac{d^n f}{dx^n}$.if n=$\frac{1}{2}$ Since then, many researchers tried to put a definition of a fractional derivative. Most of them used an *integral form* for the fractional derivative.

Two of which are the most popular ones.

1) Riemann liouville definition. For $\alpha \in [n-1, n)$ the α derivative of f is

$$D_a{}^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

2) Coputo definition. For $\alpha \in [n-1, n)$ the α derivative of f is

$$D_a{}^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}} dx$$

Now, all definitions including (i) and (ii) above satisfy the property that the fractional derivative is linear. This is the only property inherited from the first derivative by all of the definitions. However,

II. METHODS AND MATERIAL

(i) The Riemann–Liouville derivative *does not* satisfy ${D_a}^{\alpha}(1) = 0$ (${D_a}^{\alpha}(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives *do not* satisfy the known formula of the derivative of the product of two functions:

$$ {D_a}^{\alpha}(fg) = f{D_a}^{\alpha}(g) + g{D_a}^{\alpha}(f) $$

iii) All fractional derivatives *do not* satisfy the known formula of the derivative of the quotient of two functions:

$$ {D_a}^{\alpha}(f/g) = \frac{g{D_a}^{\alpha}(f) - f{D_a}^{\alpha}(g)}{g^2} $$

(iv) All fractional derivatives *do not* satisfy the chain rule:

$$ {D_a}^{\alpha}(f \circ g) = f^{\alpha}(g(t))g^{\alpha}(t) $$

(v) All fractional derivatives *do not satisfy*: $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$ general.

(vi) The Caputo definition *assumes that the function f is differentiable*.

Let f: $[0, \infty) \to$ R and $t > 0$. Then the definition of the derivative of f at t is

$$ \frac{df}{dt} = \lim_{\epsilon\to 0} \frac{f\,(t+\varepsilon)-f\,(t)}{\epsilon} . $$

According to this, one has $\frac{dt^n}{dt} = nt^{n-1}$. So the question is: Can one put a similar definition for the fractional derivative of order α, where $0 < \alpha \le 1$? Or in general for $\alpha \in (n, n + 1]$ where $n \in$ N.

Let us write T_α to denote the operator which is called the fractional derivative of order α. For $\alpha = 1$, $T1$ satisfies the following properties:

(i) $T_1(af + bg) = aT_1(g) + bT_1(f)$, for all $a, b \in$ R and f, g in the domain of T_1.

(ii) $T_1(t^p) = pt^{p-1}$

(iii) $T_1(fg) = fT_1(g) + gT_1(f)$

(iv)

$$T_1(f/g) = \frac{gT_1(f) - fT_1(g)}{g^2}$$

(v) $T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Now, we present our new definition, which is the simplest and most natural and efficient definition of fractional derivative of order $\alpha \in (0, 1]$. We should remark that the definition can be generalized to include any α. However, the case $\alpha \in (0, 1]$ is the most important one, and once it is established, the other cases are simple.

III. RESULTS AND DISCUSSION

New approach of fractional derivative/Another definition of fractional derivative

Definition 3.1. Given a function $f : [0, \infty) \longrightarrow$ R. Then the ''*conformable fractional derivative*'' of f of order α is defined by

$$\lim_{\varepsilon \to 0} \frac{f(t + (\varepsilon t)^{1-\alpha}) - f(t)}{\varepsilon^{1-\alpha}}$$

This definition satisfy all the above five properties of T_1 of derivative.

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^\alpha(t)$ exists then define $\lim_{t \to 0^+} f^\alpha(t) = f^\alpha(0)$

We will, sometimes $f^\alpha(t)$ for $T_\alpha f(t)$ write to denote the conformable fractiona derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

We should remark that $T_\alpha(t^p) = pt^{p-\alpha}$. Further, our definition coincides with the classical definitions of R–L and of Caputo on polynomials (up to a constant multiple).

As a consequence of the above definition, we obtain the following

Theorem 3.1. *If a function* $f : [0, \infty) \longrightarrow$ R *is* α*-differentiable at* $t_0 > 0 >$, $\alpha \in (0, 1]$, *then* f *is continuous at* t_0 .

Proof.

Since $f(t_0 + (\varepsilon t)^{1-\alpha}) - f(t_0) = \frac{f(t_0+(\varepsilon t)^{1-\alpha}) - f(t_0)}{\varepsilon} . \varepsilon$ then

$$\lim_{\varepsilon \to 0} f(t_0 + (\varepsilon t)^{1-\alpha}) - f(t_0) = \lim_{\varepsilon \to 0} \frac{f(t_0 + (\varepsilon t)^{1-\alpha}) - f(t_0)}{\varepsilon} . \lim_{\varepsilon \to 0} \varepsilon$$

$$\lim_{\varepsilon \to 0} f\,(t_0 + (\varepsilon t)^{1-\alpha}) - f(t_0) = f^\alpha(t_0). 0$$

$$\lim_{\varepsilon\to 0} f\,(t_0 + (\varepsilon t)^{1-\alpha}) - f(t_0) = .0$$

Which implies that

$$\lim_{\varepsilon\to 0} f\,(t_0 + (\varepsilon t)^{1-\alpha}) = f(t_0)$$

Hence f is continuous.

Theorem.3.2. *Let α ∈ (0, 1] and f , g be α-differentiable at a point t > 0. Then*
(i) $T_\alpha(af + bg) = aT_\alpha(g) + bT_\alpha(f)$, for all *a, b* ∈ R and *f , g* in the domain of T_1.
(ii) $T_\alpha(t^p) = pt^{p-1}$ for all p∈ *R*
(iii) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$
(iv)

$$T_\alpha(f/g) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$$

(v) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
VI) If in addition , f is differentiable then$T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}t$
we will prove II, III VI other follows directly from definition.
VI)ans.

$$T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}t$$

$$\lim_{\varepsilon\to 0}\frac{f(t + (\varepsilon t)^{1-\alpha}) - f(t)}{\varepsilon^{1-\alpha}}$$

Let $=h = (\varepsilon t)^{1-\alpha} = \varepsilon^{1-\alpha}t^{1-\alpha}$ *implies* $\varepsilon^{1-\alpha} = ht^{\alpha-1}$ *as* $\varepsilon \to 0, h \to 0$

$$= \lim_{h\to 0}\frac{f(t+h) - f(t)}{ht^{\alpha-1}}$$
$$= \lim_{h\to 0}\frac{f(t+h) - f(t)}{ht^{\alpha-1}}$$
$$= t^{1-\alpha}\frac{df}{dt}t.$$

II)

$$T_\alpha(t^p) = \lim_{\varepsilon\to 0}\frac{(t+(\varepsilon t)^{1-\alpha})^p - t^p}{\varepsilon^{1-\alpha}}$$
$$=$$
$$\lim_{\varepsilon\to 0}\frac{t^p + \;\; pt^{p-1}\varepsilon^{1-\alpha}t^{1-\alpha} + \;\frac{P(P-1)}{2}t^{p-2}\varepsilon^{(1-\alpha)^2}t^{(1-\alpha)^2} + \cdots \ldots\ldots\ldots\ldots + -t^p}{\varepsilon^{1-\alpha}}$$

$$= \lim_{\varepsilon\to 0}\frac{pt^{p-1}\varepsilon^{1-\alpha}t^{1-\alpha} + \;\; \frac{P(P-1)}{2}t^{p-2}\varepsilon^{(1-\alpha)^2}t^{(1-\alpha)^2} + \cdots \ldots\ldots\ldots}{\varepsilon^{1-\alpha}}$$

$$= \lim_{\varepsilon \to 0} pt^{p-1}t^{1-\alpha} + \frac{\frac{P(P-1)}{2} t^{p-2}\varepsilon^{(1-\alpha)^2} t^{(1-\alpha)^2} + \cdots \ldots \ldots \ldots}{\varepsilon^{1-\alpha}}$$

$$= pt^{p-\alpha}$$ ■

III)

$$T_\alpha(fg)(t) = \lim_{\varepsilon \to 0} \frac{f(t+(\varepsilon t)^{1-\alpha})g(t+(\varepsilon t)^{1-\alpha}) - f(t)g(t)}{\varepsilon^{1-\alpha}}$$

$$= \lim_{\varepsilon \to 0} \frac{f(t+(\varepsilon t)^{1-\alpha})g(t+(\varepsilon t)^{1-\alpha}) - f(t)g(t+(\varepsilon t)^{1-\alpha} + f(t)g(t+(\varepsilon t)^{1-\alpha} - f(t)g(t)}{\varepsilon^{1-\alpha}}$$

$$= \lim_{\varepsilon \to 0} \left(\frac{f(t+(\varepsilon t)^{1-\alpha}) - f(t)}{\varepsilon^{1-\alpha}}\right). g(t+(\varepsilon t)^{1-\alpha}) += \lim_{\varepsilon \to 0} \left(\frac{g(t+(\varepsilon t)^{1-\alpha}) - g(t)}{\varepsilon^{1-\alpha}}\right). f(t)$$

$$== T_\alpha(f) \lim_{\varepsilon \to 0} g(t+(\varepsilon t)^{1-\alpha}) + f(t)\, T_\alpha(g)$$

Since g is continuous $\lim_{\varepsilon \to 0} g(t+(\varepsilon t)^{1-\alpha}) = g(t).$ ■

Conformable fractional derivatives of certain functions.

1. $T_\alpha(t^p) = pt^{p-1}$ for all p$\in R$
2. $T_\alpha(1) = 0$
3. $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}, c \in R$
4. $T_\alpha(sinbx) = bx^{1-\alpha}cosbx, b \in R$
5. $T_\alpha(cosbx) = -bx^{1-\alpha}sinbx, b \in R$
6. $t_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$

   1. *Is proved in above thm we will prove 3, and 4 other follow directly from the definition.*

$$T_\alpha(e^{cx}) = \lim_{\varepsilon \to 0} \frac{e^{(cx + c(\varepsilon x)^{1-\alpha})} - e^{cx}}{\varepsilon^{1-\alpha}}$$

$$T_\alpha(e^{cx}) = \lim_{\varepsilon \to 0} \frac{e^{cx}e^{c(\varepsilon x)^{1-\alpha}} - e^{cx}}{\varepsilon^{1-\alpha}}$$

Here h=$h = (\varepsilon x)^{1-\alpha} \Rightarrow \varepsilon^{1-\alpha} = h.x^{\alpha-1}$

Putting the values

$$T_\alpha(e^{cx}) = \lim_{\varepsilon\to 0} \frac{e^{cx}(e^{ch}-1)}{h.x^{\alpha-1}}$$

$$T_\alpha(e^{cx}) = \lim_{h\to 0} \frac{e^{cx}(e^{ch}-1).c}{c.h.x^{\alpha-1}}$$

$$T_\alpha(e^{cx}) = \frac{e^{cx}c}{x^{\alpha-1}} \lim_{h\to 0} \frac{(e^{ch}-1).}{c.h.}$$

$$as\ \varepsilon \to 0, \Rightarrow h \to 0$$

$$T_\alpha(e^{cx}) = \frac{e^{cx}c}{x^{\alpha-1}}.1$$

$T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}, c \in R$. ■

4)

$$T_\alpha(sinbx) = bx^{1-\alpha}cosbx, b \in R$$

By using definition.

$$= \lim_{\varepsilon\to 0} \frac{sinb(x+(\varepsilon x)^{1-\alpha}) - sinbx}{\varepsilon^{1-\alpha}}$$

$$= \lim_{\varepsilon\to 0} \frac{2cosb\left(x+\frac{(\varepsilon x)^{1-\alpha}}{2}\right) sin\frac{b(\varepsilon x)^{1-\alpha}}{2}}{\varepsilon^{1-\alpha}}$$

$$= cosbx \lim_{\varepsilon\to 0} \frac{sin\frac{b(\varepsilon x)^{1-\alpha}}{2}}{\frac{b(x\varepsilon)^{1-\alpha}}{2}}.bx^{1-\alpha}$$

$= cosbx.\, bx^{1-\alpha}$ ■

In similar mannar as above $T_\alpha(cosbx) = -bx^{1-\alpha}sinbx, b \in R$ *can be proved.*

However, it is worth noting the following conformable fractional derivatives of certain functions:

7. $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = \left(e^{\frac{1}{\alpha}t^\alpha}\right)$
8. $T_\alpha\left(sin\frac{1}{\alpha}t^\alpha\right) = \left(sin\frac{1}{\alpha}t^\alpha\right)$
9. $T_\alpha\left(cos\frac{1}{\alpha}t^\alpha\right) = \left(cos\frac{1}{\alpha}t^\alpha\right)$

This result can proved as above.by using definition.

IV. CONCLUSION

one can also generalize this derivative.

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