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Feynman Kernel in Fractional Quantum Systems

Omprakash Atale
Department Physics, Khandesh College Education Society's Moolji Jaitha College
"An Autonomous College affiliated to K.B.C. North Maharashtra University, Jalgaon, India"

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ABSTRACT
In this paper, we have sketched what is to be known as fractional path integral representation in quantum mechanics. We will begin with fractional Schrödinger's equation in the framework of Caputo fractional derivatives. Furthermore, Feynman kernel for derived path integrals is established.

Keywords : Fractional Quantum Mechanics, Feynman Kernel, Caputo Derivative

I. INTRODUCTION

Nearly a decade ago, a group of researchers proposed the fractional form of Schrödinger's equation in the framework of both Riemann-Liouville (RL) and Caputo fractional derivatives [1]. The aim of this short paper is to establish the path integral formulation using the fractional form of Schrödinger's equation in the framework of Caputo derivative. The main reason why we have chosen to work on Caputo derivative instead of Riemann-Liouville derivative is because it makes more physical sense in the terms of both applications and understanding of the subject. The fractional differential equations that are formulated in the framework of Caputo derivatives require only boundary conditions; this coincides with the needs of boundary conditions in quantum mechanics. Whereas, on the other hand, Riemann-Liouville derivative needs initial condition. Many other types of Schrödinger's equation are also established depending on the framework of different kinds of fractional derivative, for more information, refer to references [2]-[4].

Consider a function depending on n variables $x_1, x_2......x_n$ over the domain $\Omega = [a_1, b_1] \times ... \times [a_n, b_n]$, then for $0 < \alpha_k < 1$, where α_k is the order of the derivative, we define left and right partial Riemann-Liouville (RL) and Caputo fractional derivatives as follows [1]:

$$({}_{+}\partial_k^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \partial x_k \int_{a_k}^{x_k} \frac{f(x_1, ..., x_{k-1}, u, x_{k+1}, ..., x_n)}{(x_k - u)^{\alpha_k}} du, \qquad 1.1$$

$$({}_{-}\partial_k^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha_k)} \partial x_k \int_{x_k}^{b_k} \frac{f(x_1,\dots,x_{k-1},u,x_{k+1},\dots,x_n)}{(u-x_k)^{\alpha_k}} du, \quad 1.2$$

$$({}_{+}^{C}\partial_k^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \int_{a_k}^{x_k} \frac{\partial_u f(x_1,\dots,x_{k-1},u,x_{k+1},\dots,x_n)}{(x_k-u)^{\alpha_k}} du, \quad 1.3$$

and

$$({}_{-}^{C}\partial_k^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha_k)} \int_{x_k}^{b_k} \frac{\partial_u f(x_1,\dots,x_{k-1},u,x_{k+1},\dots,x_n)}{(u-x_k)^{\alpha_k}} du, \quad 1.4$$

respectively. Here, ∂x_k denotes the partial derivative with respect to x_k, The subscript k and the superscript α indicate that the derivative is taken with respect to x_k and it is of order α_k $(\alpha_k = \alpha)$.

II. FRACTONAL SCHRODINGER'S EQUATION A TIME EVOLUTION OPERATOR

From [1], Eqn. (13), we have the following fractional Schrödinger's equation

$$i\hbar^\alpha ({}_{+}^{C}\partial_0^\alpha)\Psi = \frac{-\hbar^{2\alpha}}{2m^\alpha}({}_{+}^{C}\partial_k^\alpha)({}_{+}^{C}\partial_k^\alpha)\Psi + V\Psi \quad 2.1$$

where the subscript 0 in $({}_{+}^{C}\partial_0^\alpha)$ denotes that fractional Caputo derivative has been taken with respect to time, and similarly, the subscript k in $({}_{+}^{C}\partial_k^\alpha)$ denotes that the derivative has been taken with respect to the spatial dimensions. Eqn. (2.1) can be written into the following form for any arbitrary state $|\Psi(t)\rangle$.

$$i\hbar^\alpha ({}_{+}^{C}\partial_0^\alpha)|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle \quad 2.2$$

with

$$\hat{H} = \frac{-\hbar^{2\alpha}}{2m^\alpha}({}_{+}^{C}\partial_k^\alpha)({}_{+}^{C}\partial_k^\alpha) + V. \quad 2.3$$

With the above definitions in mind, authors in [5] have provided a time evolution of the solution for Eqn. 2.2 as

$$|\Psi(t)\rangle = U(t,t_0)|\Psi(t_0)\rangle \quad 2.4$$

with

$$i\hbar^\alpha ({}_{+}^{C}\partial_0^\alpha)]U(t,t_0) = \hat{H}U(t,t_0) \quad 2.5$$

where the time evolution operator is defined as

$$U(t,t_0) = E_\alpha\left(-\frac{i}{\hbar^\alpha}\hat{H}\Delta t^\alpha\right), \quad 2.6$$

and $E_\alpha(z)$ is the Mittag–Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}. \quad 2.7$$

One can note that the evolution operator that we have written here is slightly different from what authors have proposed in [5], this is because authors in [5] haven't raised their reduced Plank's constant to the power of the order of derivative in their Schrödinger's equation, whereas, in our equations, we have used $\hbar^{\alpha}$ instead of only $\hbar$. Since we know that for any time evolution operator satisfies the property

$$U(t_2,t_1)U(t_1,t_0)=U(t_2,t_0), \tag{2.8}$$

$$U(t_0+\Delta t,t_0)=I-\frac{i}{\hbar^{\alpha}}\hat{H}(t_1)\Delta t^{\alpha}+O\left(\left(\Delta t^{\alpha}\right)^2\right) \tag{2.9}$$

where I is the unit operator and $t_1=t_0+\Delta t$, yielding

$$U(t,t_0)=\lim_{N\to\infty}\left(I-\frac{i}{\hbar^{\alpha}}\hat{H}(t\equiv t_N)\Delta t^{\alpha}\right)\left(I-\frac{i}{\hbar^{\alpha}}\hat{H}(t\equiv t_{N-1})\Delta t^{\alpha}\right)\times\ldots\times\left(I-\frac{i}{\hbar^{\alpha}}\hat{H}(t\equiv t_1)\Delta t^{\alpha}\right) \tag{2.10}$$

where the infinitesimal time interval Δt^{α} is given by

$$\Delta t^{\alpha}=\frac{(t-t_0)^{\alpha}}{N}. \tag{2.11}$$

For time independent Hamiltonian, we have

$$U(t,t_0)=\lim_{N\to\infty}\left(I-\frac{i}{\hbar^{\alpha}}\hat{H}\Delta t^{\alpha}\right)^N=e^{-\frac{i}{\hbar^{\alpha}}\hat{H}(t-t_0)^{\alpha}}, \tag{2.12}$$

from this, we can get the Trotter's formula if the Hamiltonian can be written as $H=A+B$, which is

$$e^{-\frac{i}{\hbar^{\alpha}}(t-t_0)^{\alpha}(A+B)}=\lim_{N\to\infty}\left(e^{-\frac{i}{\hbar^{\alpha}}\Delta t^{\alpha}A}\,e^{-\frac{i}{\hbar^{\alpha}}\Delta t^{\alpha}B}\right)^N, \tag{2.13}$$

usually, A and B are kinetic and potential energy of the system respectively.

III.FEYNMAN KERNEL

We know from the definition of Hamiltonian that in order for our system to work mathematically, Hamiltonian must be momentum and position dependent both in Heisenberg's and Schrödinger's picture. Therefore, we define our fractional momentum operator and position operator as follows

$$\hat{P}_{\alpha}^{k}|\Psi\rangle=p|\Psi\rangle, \tag{3.1}$$

$$\hat{X}|\Psi\rangle=x|\Psi\rangle, \tag{3.2}$$

where

$$\hat{P}_{\alpha}^{k}=-i\hbar^{\alpha}({}_{+}^{C}\partial_{k}^{\alpha}). \tag{3.3}$$

Taking into account the following completeness conditions

$$[\hat{X},\hat{P}_{\alpha}^{k}]=i\hbar^{\alpha}\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{3.4}$$

$$[\hat{X},\hat{X}]=[\hat{P}_{\alpha}^{k},\hat{P}_{\alpha}^{k}]=0, \tag{3.5}$$

$$\begin{aligned}\hat{P}_{\alpha}^{k}|p\rangle=p|p\rangle,\quad \langle p|\hat{P}_{\alpha}^{k}=\langle p|p,\\ \hat{X}|x\rangle=x|x\rangle,\quad \langle x|\hat{X}=\langle x|x,\end{aligned} \tag{3.6}$$

$$\langle x|p\rangle=\frac{e^{ipx/\hbar^{\alpha}}}{\sqrt{2\pi\hbar^{\alpha}}}\quad \langle p|x\rangle=\frac{e^{-ipx/\hbar^{\alpha}}}{\sqrt{2\pi\hbar^{\alpha}}} \tag{3.7}$$

$$\int_{-\infty}^{+\infty} |p\rangle\langle p| dp = I, \qquad 3.8$$

$$\int_{-\infty}^{+\infty} |x\rangle\langle x| dx = I, \qquad 3.9$$

we can now write our Hamiltonian as

$$H(t) = H\left(\hat{P}_\alpha^k, \hat{X}, t\right). \qquad 3.10$$

Insert Eqn. (3.8) into (2.10) successively to find

$$U(t,t_0) = \lim_{N\to\infty} \prod_{j=0}^{N} \left(\int_{-\infty}^{+\infty} dx_j \right) |x_N\rangle\langle x_0| \prod_{j=1}^{N} \langle x_j | I - \frac{i}{\hbar^\alpha} H_j\left(\hat{P}_\alpha^k, \hat{X}\right)\Delta t^\alpha \left| x_{j-1} \right\rangle \qquad 3.11$$

where

$$H_j\left(\hat{P}_\alpha^k, \hat{X}\right) = H\left(\hat{P}_\alpha^k, \hat{X}, t_j\right) \qquad 3.12$$

and the quantity

$$K\left(x_j, t_j, x_{j-1}, t_{j-1}\right) = \langle x_j | I - \frac{i}{\hbar^\alpha} H_j\left(\hat{P}_\alpha^k, \hat{X}\right)\Delta t^\alpha \left| x_{j-1} \right\rangle \qquad 3.13$$

which appears in Eqn. (3.11) is known as the Feynman kernel. If Hamiltonian has the form

$$H\left(\hat{P}_\alpha^k, \hat{X}; t\right) = \sum_{m,n} a_{m,n}(t)\left(\hat{P}_\alpha^k\right)^m \hat{X}^n \qquad 3.14$$

then, insert Eqn. (3.8) on the left hand side of

$$\langle x_j | I - \frac{i}{\hbar^\alpha} H_j\left(\hat{P}_\alpha^k, \hat{X}\right)\Delta t^\alpha \left| x_{j-1} \right\rangle \qquad 3.15$$

in Eqn. (3.11) and then use relation (3.7) to get a valid representation of Feynman kernel. Therefore, we get

$$K^{\left(\hat{P}_\alpha^k, \hat{X}\right)}\left(x_j, t_j, x_{j-1}, t_{j-1}\right) = \int_{-\infty}^{+\infty} \frac{dx_j}{2\pi\hbar^\alpha} e^{ip_j\left(x_j - x_{j-1}\right)/\hbar^\alpha} \left(I - \frac{i}{\hbar^\alpha} H_j\left(p_j, q_{j-1}\right)\Delta t^\alpha \right) \qquad 3.16$$

where the sub-script on K represents operator ordering. We need to take into account the issue of operator ordering because it changes the argument below position in the Hamiltonian. Therefore, if the Hamiltonian has the form

$$H\left(\hat{P}_\alpha^k, \hat{X}; t\right) = \sum_{m,n} b_{m,n}(t)\hat{X}^m \left(\hat{P}_\alpha^k\right)^n \qquad 3.17$$

then, this in turn assembles Feynman kernel into the following form

$$K^{\left(\hat{X}, \hat{P}_\alpha^k\right)}\left(x_j, t_j, x_{j-1}, t_{j-1}\right) = \int_{-\infty}^{+\infty} \frac{dx_j}{2\pi\hbar^\alpha} e^{ip_j\left(x_j - x_{j-1}\right)/\hbar^\alpha} \left(I - \frac{i}{\hbar^\alpha} H_j\left(p_j, q_j\right)\Delta t^\alpha \right). \qquad 3.18$$

IV. CONCLUSION

In this paper, we have provided an expression for Feynman kernel for fractional quantum mechanical systems in the framework of Caputo fractional derivative. We have begun by stating the completeness conditions and then developing the time evolution operator to finally get in the expression for Feynman kernel. There has been an increasing interest in fractional quantum mechanics since last few decades and many fractional quantum mechanical systems are established since then [6]. The completeness conditions and the expression for Feynman kernel that we have derived can be useful in deriving the path integral formulation of fractional quantum mechanical systems. Thus, the result can therefore be applied to deriving the trace formula, coherent states, fermionic path integrals etc. in fractional quantum mechanical systems.

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