International Journal of Scientific Research in Science and Technology
Print ISSN: 2395-6011 | Online ISSN: 2395-602X (www.ijsrst.com)
doi : https://doi.org/10.32628/IJSRST229136

Quantization of $R \times S^3$ Topological Klein-Gordon Scalar Field

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Article Info
Volume 9, Issue 1
Page Number : 206-217
Publication Issue
January-February-2022
Article History
Accepted : 01 Feb 2022
Published : 11 Feb 2022

ABSTRACT
In this paper, we have portrayed scalar and complex Klein-Gordon field theory on $R \times S^3$ topological space. The corresponding Klein-Gordon equation was established by M. Carmeli in October 1983. The field theory is formulated using differential operators defined on S^3 topology instead of ordinary Cartesian operators. Furthermore, we have quantized the theory and commutation relations along with the Hamiltonian for the theory are derived.

Keywords: Klein-Gordon scalar field, Quantization, $R \times S^3$ topology.

I. INTRODUCTION

Klein-Gordon field theory has been of grave importance in modern particle physics for describing the dynamics of spin zero particles. A successful attempt of describing field theories on $R \times S^3$ topology was done by M. Carmeli and A. Malka in a series of 6 papers [1]-[6]. The $R \times S^3$ topological Klein-Gordon equation was first introduced in [1] and a more general solution using group theoretic method was established by authors. However, a rigorous and applicable treatment in the framework of quantum field theory for those equations using the plane wave solution haven't been yet formulated on $R \times S^3$ topology. The main aim of this paper is to establish Klein-Gordon field theory on $R \times S^3$ topology where we have sketched a classic formulation and it's applications in quantum field theory. From [1], we have the following form of Klein-Gordon equation on $R \times S^3$ topology.

$$\left(L^2 - \frac{1}{\gamma^2}\frac{\partial^2}{\partial t^2}\right)\phi(t,\theta) = \left(\frac{I_0\gamma}{\hbar}\right)^2 \phi(t,\theta). \tag{1.1}$$

Here, $\theta = (\theta^1, \theta^2, \theta^3)$ are three rotational angles such as Euler angles and $L = (L_1, L_2, L_3) = (L_x, L_y, L_z)$ is the corresponding differential operator given by

$$L_1 = \frac{-\sin\theta^3}{\sin\theta^2}\frac{\partial}{\partial\theta^1} - \cos\theta^3\frac{\partial}{\partial\theta^2} + \cot\theta^3\sin\theta^3\frac{\partial}{\partial\theta^3}, \tag{1.2}$$

$$L_2 = \frac{-\cos\theta^3}{\sin\theta^2}\frac{\partial}{\partial\theta^1} - \sin\theta^3\frac{\partial}{\partial\theta^2} + \cot\theta^3\cos\theta^3\frac{\partial}{\partial\theta^3}, \tag{1.3}$$

$$L_3 = -\frac{\partial}{\partial\theta^3}. \tag{1.4}$$

The operator $L^2 = L_1^2 + L_2^2 + L_3^2 = L_x^2 + L_y^2 + L_z^2$ is then give by

$$L^2 = \frac{1}{\sin\theta^2}\frac{\partial}{\partial\theta^2}\left(\sin\theta^2\frac{\partial}{\partial\theta^2}\right) + \frac{1}{\sin^2\theta^2}\left(\frac{\partial^2}{\partial\theta^{1^2}} - 2\cos\theta^2\frac{\partial^2}{\partial\theta^1\,\partial\theta^3} + \frac{\partial^2}{\partial\theta^{3^2}}\right). \quad (1.5)$$

Plane wave solution of Eqn. (1.1) is given by

$$\phi(\theta) = e^{(i/\hbar)(Et - J.\theta)} \quad (1.6)$$

where $\theta^\alpha = (\theta^0, \theta^1, \theta^2, \theta^3)$, $\theta^0 = ct$ and

$$J^\alpha = \left(J^0, J^k\right) = (E, \gamma J), \quad (1.7)$$

$$J_\alpha = (J_0, J_k) = (E, -\gamma J) \quad (1.8)$$

is the angular momentum four vector with $J = \left(J_x, J_y, J_z\right)$ and $J_k = i\hbar\gamma L_k$. Furthermore, using the definition of J^α, J_α and $\gamma = c(m_0/I_0)^{1/2}$ we can conclude the following relation:

$$E_J = J^0 = J_0 = \pm(\gamma^2 J^2 + I_0^2\gamma^4)^{1/2}, \quad (1.9)$$

$$J^\alpha J_\alpha = E_J^2 - \gamma^2 J^2 = I_0^2\gamma^4. \quad (1.10)$$

Here, m_0 and I_0 are rest mass and moment of inertia respectively. Another more general solution of Eqn. (1.1) using group theoretic methods can be found in [1].

II. FIELD THEORY AND QUANTIZATION

Let

$$L^\mu = \frac{\partial}{\partial\theta_\mu} = \left(\frac{1}{\gamma}\frac{\partial}{\partial t}, -L\right), \quad (2.1)$$

$$L_\mu = \frac{\partial}{\partial\theta^\mu} = \left(\frac{1}{\gamma}\frac{\partial}{\partial t}, L\right), \quad (2.2)$$

and

$$L^\mu L_\mu = \frac{1}{\gamma^2}\frac{\partial^2}{\partial t^2} - L^2. \quad (2.3)$$

Now, we can construct a Lagrangian for $R \times S^3$ topological Klein-Gordon equation as

$$\mathcal{L} = \frac{1}{2}L_\mu\phi L^\mu\phi - \frac{m_0^2}{2}\phi^2 \quad (2.4)$$

where we have employed $\hbar = c = 1$. For constructing a variational principle on $R \times S^3$ topology, we first assume that our Lagrangian depends on fields ϕ and their derivatives $L_\mu\phi$. Hence, $\mathcal{L} = \mathcal{L}\left(\phi, L_\mu\phi\right)$ and the action has the form

$$S(\Omega) = \int_\Omega L\left(\phi, L_\mu\phi\right)d^4\theta \quad (2.5)$$

whose variation $\delta S(\Omega) = 0$ would lead us to the following Euler-Lagrange's equation:

$$\frac{\partial L}{\partial\phi} - L_\mu\left(\frac{\partial L}{\partial\left(L_\mu\phi\right)}\right) = 0. \quad (2.6)$$

A detailed derivation of the above equation can be found in [6]. We can now plugin Eqn. (2.4) in (2.6) to get the following form of Klein-Gordon equation:

$$\left(L_\mu L^\mu + m_0^2\right)\phi(\theta) = 0 \tag{2.7}$$

The process for quantization of $R \times S^3$ topological Klein-Godon field theory will be similar to that of quantization in classical mechanics. First, we define canonically conjugate momentum to construct Hamiltonian and then solve corresponding commutation relations.

Define a momentum canonically conjugate to the field variable $\phi(\theta)$ as follows:

$$\Pi(\theta) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\theta)} \tag{2.8}$$

from which we can construct a Hamiltonian density as

$$\mathcal{H} = \Pi(\theta)\dot{\phi}(\theta) - \mathcal{L} \tag{2.9}$$

which yields a Hamiltonian of the form

$$H = \int d^3\theta \mathcal{H} = \int d^3\theta\left(\Pi(\theta)\dot{\phi}(\theta) - \mathcal{L}\right). \tag{2.10}$$

Note that Lagrangian density (2.4) can be written as

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}L\phi.L\phi - \frac{m_0^2}{2}\phi^2. \tag{2.11}$$

This leads us to the following Hamiltonian density for the system

$$\mathcal{H} = \Pi(\theta)\dot{\phi}(\theta) - \mathcal{L} \tag{2.12}$$

$$= \Pi(\theta)\dot{\phi}(\theta) - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}L\phi.L\phi + \frac{m_0^2}{2}\phi^2 \tag{2.13}$$

$$= \Pi(\theta)\Pi(\theta) - \frac{1}{2}\Pi(\theta)^2 + \frac{1}{2}L\phi.L\phi + \frac{m_0^2}{2}\phi^2 \tag{2.14}$$

$$= \frac{1}{2}\Pi(\theta)^2 + \frac{1}{2}L\phi.L\phi + \frac{m_0^2}{2}\phi^2. \tag{2.15}$$

Therefore, using Eqn. (2.10), we get a Hamiltonian for our theory as

$$H = \int d^3\theta\left(\frac{1}{2}\Pi(\theta)^2 + \frac{1}{2}L\phi.L\phi + \frac{m_0^2}{2}\phi^2\right). \tag{2.16}$$

Assuming equal time canonical Poisson brackets relations between $\Pi(\theta)$ and $\phi(\theta)$ to be

$$\left\{\phi(\theta), \phi(\tilde{\theta})\right\}_{\theta^0=\tilde{\theta}^0} = \left\{\Pi(\theta), \Pi(\tilde{\theta})\right\}_{\theta^0=\tilde{\theta}^0} = 0 \tag{2.17}$$

$$\left\{\phi(\theta), \Pi(\tilde{\theta})\right\}_{\theta^0=\tilde{\theta}^0} = \delta^3\left(\theta - \tilde{\theta}\right) \tag{2.18}$$

then, we can easily show that the dynamical equations of first order in the Hamiltonian form can be written as

$$\dot{\phi}(\theta) = \{\phi(\theta), H\} \tag{2.19}$$

$$\dot{\Pi}(\theta) = \{\Pi(\theta), H\} \tag{2.20}$$

These relations can explicitly be formulated as follows. Recall the definition of Hamiltonian from Eqn. (2.6) and then using Eqn. (2.19), we can write

$$\dot{\phi}(\theta) = \{\phi(\theta), H\}$$

(2.21)

$$=\left\{\phi(\theta),\int d^3\tilde{\theta}\left(\frac{1}{2}\Pi^2(\tilde{\theta})+\frac{1}{2}L_{\tilde{\theta}}\phi.L_{\tilde{\theta}}\phi+\frac{m_0^2}{2}\phi^2(\tilde{\theta})\right)\right\}_{\theta^0=\tilde{\theta}^0} \quad (2.21)$$

$$=\frac{1}{2}\int d^3\tilde{\theta}\{\phi(\theta),\Pi^2(\tilde{\theta})\}_{\theta^0=\tilde{\theta}^0}=\Pi(\theta).$$

(2.22)

Similarly,

$$\dot{\Pi}(\theta)=\{\Pi(\theta),H\}$$

(2.23)

$$=\left\{\Pi(\theta),\int d^3\tilde{\theta}\left(\frac{1}{2}\Pi^2(\tilde{\theta})+\frac{1}{2}L_{\tilde{\theta}}\phi.L_{\tilde{\theta}}\phi+\frac{m_0^2}{2}\phi^2(\tilde{\theta})\right)\right\}_{\theta^0=\tilde{\theta}^0} \quad (2.24)$$

$$=\int d^3\tilde{\theta}\left[L_{\tilde{\theta}}\phi(\tilde{\theta}).\{\Pi(\theta),L_{\tilde{\theta}}\phi(\tilde{\theta})\}+m_0^2\phi(\tilde{\theta})\{\Pi(\theta),\phi(\tilde{\theta})\}\right]_{\theta^0=\tilde{\theta}^0}$$

(2.25)

$$=L.L\phi(\theta)-m_0^2\phi(\theta).$$

(2.26)

Therefore, we get

$$\dot{\phi}(\theta)=\Pi(\theta)$$

(2.27)

and

$$\dot{\Pi}(\theta)=L^2\phi(\theta)-m_0^2\phi(\theta).$$

(2.28)

Taking both sides derivative of Eqn. (2.29) and using Eqn. (2.30) we obtain what is kown as second order equation which eventually gives Kelin-Gordon equation in it's original form on $R\times S^3$ topology, that is, Eqn. (2.7).

III.SOLUTION AND CREATION-ANNIHIIATION OPERATORS

We know that plane wave solution to $R\times S^3$ topological equation is given by Eq. (1.6). Therefore, using this information, we can construct a corresponding general solution in terms of the corresponding plane wave solution as

$$\phi(\theta)=\int\frac{d^3J}{\sqrt{(2\pi)^3 2J^0}}\left(e^{-iJ.\theta}a(J)+e^{iJ.\theta}a^\dagger(J)\right) \quad (3.1)$$

where

$$a(J)=\frac{a(J)}{\sqrt{2J^0}} \quad (3.2)$$

and

$$a^\dagger(J)=\frac{a^\dagger(J)}{\sqrt{2J^0}} \quad (3.3)$$

are functions that later act annihilation and creation operators later. Therefore, as now we have constructed our general solution, using Eqn. (2.29) we can get the conjugate momentum as follows:

$$\Pi(\theta)=\dot{\phi}(\theta)=-i\int d^3J\sqrt{\frac{J^0}{2(2\pi)^3}}\left(e^{-iJ.\theta}a(J)-e^{iJ.\theta}a^\dagger(J)\right). \quad (3.4)$$

Since definition (3.1) ad (3.4) are invertible, we can construct construct the following representation of functions (3.2) and (3.3):

$$a(J) = \frac{1}{\sqrt{(2\pi)^3 2J^0}} \int d^3\theta e^{iJ.\theta}\left(J^0\phi(\theta) + i\Pi(\theta)\right) \tag{3.5}$$

$$= \frac{1}{\sqrt{(2\pi)^3 2J^0}} \int d^3\theta e^{iJ.\theta} \leftrightarrow \partial_t\phi(\theta). \tag{3.6}$$

$$a^\dagger(J) = \frac{1}{\sqrt{(2\pi)^3 2J^0}} \int d^3\theta e^{-iJ.\theta}\left(J^0\phi(\theta) - i\Pi(\theta)\right) \tag{3.7}$$

$$= \frac{-1}{\sqrt{(2\pi)^3 2J^0}} \int d^3\theta e^{-iJ.\theta} \leftrightarrow \partial_t\phi(\theta). \tag{3.8}$$

Since now we have obtained a reasonable representation for solution and operators, we can now apply this to quantize our theory. Therefore, the commutation relations for (3.1) and (3.4) reads:

$$\begin{aligned}
&[\phi(\theta), \phi(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} \\
&= \int\int \frac{d^3J}{\sqrt{(2\pi)^3 2J^0}} \frac{d^3J'}{\sqrt{(2\pi)^3 2J'^0}} \\
&\times \left(e^{-iJ.\theta - iJ'.\theta}[a(J), a(J')] + e^{-iJ.\theta + iJ'.\theta}[a(J), a^\dagger(J')]\right. \\
&\left. + e^{iJ.\theta - iJ'.\theta}[a^\dagger(J), a(J')] + e^{iJ.\theta + iJ'.\theta}[a^\dagger(J), a^\dagger(J')]\right) = 0
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&[\Pi(\theta), \Pi(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} \\
&= -\int\int d^3J d^3J'^{\frac{\sqrt{J^0J'^0}}{2(2\pi)^3}} \\
&\times \left(e^{-iJ.\theta - iJ'.\theta}[a(J), a(J')] - e^{-iJ.\theta + iJ'.\theta}[a(J), a^\dagger(J')]\right. \\
&\left. - e^{iJ.\theta - iJ'.\theta}[a^\dagger(J), a(J')] + e^{iJ.\theta + iJ'.\theta}[a^\dagger(J), a^\dagger(J')]\right) = 0
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&[\phi(\theta), \Pi(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} \\
&= -\frac{i}{(2\pi)^3}\int\int d^3J d^3J' \sqrt{\frac{J'^0}{4J^0}} \times \left(e^{-iJ.\theta - iJ'.\theta}[a(J), a(J')]\right. \\
&- e^{-iJ.\theta + iJ'.\theta}[a(J), a^\dagger(J')] + e^{iJ.\theta - iJ'.\theta}[a^\dagger(J), a(J')] \\
&\left. - e^{iJ.\theta + iJ'.\theta}[a^\dagger(J), a^\dagger(J')]\right) = i\delta^3(\theta - \tilde{\theta}).
\end{aligned} \tag{3.11}$$

In a similar manner, through a process of long and pain full calculations, we can derive similar commutation relations for operators (3.2) and (3.3) using relation (3.7) and (3.7). Therefore, we get

$$[a(J), a(J')] = [a^\dagger(J), a^\dagger(J')] = 0 \tag{3.12}$$

$$[a(J), a^\dagger(J')] = \delta^3(J - J') \tag{3.13}$$

To understand the physical meaning of this operators and their working, let us take a look at the Hamiltonian of our system. Therefore, from Eqn.(2.12), we have

$$H = \frac{1}{2}\int d^3\theta \left(\Pi^2(\theta) + L\phi.L\phi + m_0^2\phi^2(\theta)\right). \tag{3.14}$$

We can break our Hamiltonian into three pieces in order to simplify calculations as follows:

$$H_1 = \frac{1}{2}\int d^3\theta \Pi^2(\theta)$$

(3.15)

$$H_2 = \frac{1}{2}\int d^3\theta L\phi . L\phi \quad (3.16)$$

$$H_3 = \frac{1}{2}\int d^3\theta m_0^2\phi^2(\theta). \quad (3.17)$$

Using definition (3.1) and (3.4), we get the following values of H_1, H_2 and H_3

$$H_1 = -\frac{1}{2}\int d^3JJ^0(e^{-2iJ^0\theta^0}a(J)a(-J) - a(J)a^\dagger(J) - a^\dagger(J)a(J) + e^{2iJ^0\theta^0}a^\dagger(J)a^\dagger(-J)) \quad (3.18)$$

$$H_2 = -\frac{1}{2}\int d^3J\frac{J^2}{J^0}(-e^{-2iJ^0\theta^0}a(J)a(-J) - a(J)a^\dagger(J) - a^\dagger(J)a(J) - e^{2iJ^0\theta^0}a^\dagger(J)a^\dagger(-J)) \quad (3.19)$$

$$H_3 = \frac{1}{2}\int d^3J\frac{1}{J^0}(e^{-2iJ^0x^0}a(J)a(-J) + a(J)a^\dagger(J) + a^\dagger(J)a(J) + e^{-2iJ^0x^0}a^\dagger(J)a^\dagger(-J)) \quad (3.20)$$

respectively. Therefore, adding those terms we get

$$H = \frac{1}{2}\int d^3JJ^0\left(a(J)a^\dagger(J) + a^\dagger(J)a(J)\right). \quad (3.21)$$

Using relation (1.9), we can write the above Hamiltonian as

$$H = \int d^3J\frac{E_J}{2}\left(a(J)a^\dagger(J) + a^\dagger(J)a(J)\right). \quad (3.22)$$

It follows now that

$$[a(J), H] = Ea(J) \quad (3.23)$$

$$\left[a^\dagger(J), H\right] = -Ea^\dagger(J) \quad (3.24)$$

which shows that operators $a(J)$ and $a^\dagger(J)$ annihilate and create a quantum of energy.

IV. NORMAL ORDERING AND NUMBER OPERATOR

We know since the study of quantum mechanics that the ordering of the operators is ambiguous and it affects calculations. To remove this ambiguity, we define normal ordering where creation operators stand to the left of annihilation operators. Thus, if we normal order our Hamiltonian, we get

$$H^{N.O} = \int d^3JE_Ja^\dagger(J)a(J) = \int d^3JE_JN(J) \quad (4.1)$$

where $N(J) = a^\dagger(J)a(J)$ is the number operator. From this,the total number operator for the system can be defined as

$$N = \int d^3JN(J) = \int d^3Ja^\dagger(J)a(J). \quad (4.2)$$

It now follows from the definition of number operator that

$$[a(J), N(J')] = \left[a(J), a^\dagger(J')a(J')\right] \quad (4.3)$$

$$= \left[a(J), a^{\dagger}(J')\right]a(J') \tag{4.4}$$

$$= a(J')\delta^3(J - J') \tag{4.5}$$

and

$$\left[a^{\dagger}(J), N(J')\right] = \left[a^{\dagger}(J), a^{\dagger}(J')a(J')\right] \tag{4.6}$$

$$= a^{\dagger}(J')\left[a^{\dagger}(J), a(J')\right] \tag{4.7}$$

$$= -a^{\dagger}(J')\delta^3(J - J'). \tag{4.8}$$

Thus,

$$[a(J), N] = \left[a(J), \int d^3J'N(J')\right] \tag{4.9}$$

$$= \int d^3J'\left(a(J')\delta^3(J - J')\right) \tag{4.10}$$

$$= a(J) \tag{4.11}$$

and

$$\left[a^{\dagger}(J), N\right] = \left[a^{\dagger}(J), \int d^3J'N(J')\right] \tag{4.12}$$

$$= \int d^3J'\left(-a^{\dagger}(J')\delta^3(J - J')\right) \tag{4.13}$$

$$= -a^{\dagger}(J). \tag{4.14}$$

The above calculation is just another way to show that $a^{\dagger}(J)$ and $a(J)$ raise and lower the number of quanta by one unit.

V. ENERGY EIGENSTATES

Consider the normal ordered Hamiltonian

$$H = \int d^3JE_Ja^{\dagger}(J)a(J). \tag{5.1}$$

The energy eigenstates of this Hamiltonian is

$$H|E\rangle = E|E\rangle. \tag{5.2}$$

Thus,

$$E = \langle E|H|E\rangle \tag{5.3}$$

$$= \langle E|\int d^3JE_Ja^{\dagger}(J)a(J)|E\rangle \tag{5.3}$$

$$= \int d^3JE_J\langle E|a^{\dagger}(J)a(J)|E\rangle \geq 0. \tag{5.5}$$

This yields that $E \geq 0$ and we do not have to worry about negative energy states. Using relation (3.26) and (3.27), we have

$$[a(J), H]|E\rangle = E_J a(J)|E\rangle \tag{5.6}$$

$$a(J)H|E\rangle - Ha(J)|E\rangle = E_J a(J)|E\rangle \tag{5.7}$$

$$H\{a(J)|E\rangle\} = \left(E - E_J\right)\{a(J)|E\rangle\}. \tag{5.8}$$

In a similar manner, we get

$$H\left\{a^{\dagger}(J)|E\rangle\right\} = \left(E + E_J\right)\left\{a^{\dagger}(J)|E\rangle\right\}. \tag{5.9}$$

Since a(J) acts as an annihilation operator, their must exist a state with minimum energy due to relation (5.6). Thus

$$a(J)|E_{min}\rangle = 0. \tag{5.10}$$

Beyond this, we cannot lower our energy states further . This minimum energy state is known as vacuum state and can also be denoted by $|0\rangle$.

VI. GREEN'S FUNCTION

Green's function is fundamental in quantum field theory in studying the solutions of inhomogeneous differential equations where interaction between fields or particles takes place. The simplest case of Klein-Gordon field with an external source $J(\theta)$ is

$$\left(L_\mu L^\mu + m_0^2\right)\phi(\theta) = J(\theta) \tag{6.1}$$

and the corresponding Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} L_\mu \phi L^\mu \phi - \frac{m_0^2}{2}\phi^2 + J\phi. \tag{6.2}$$

The Green's function for a given inhomogeneous equation is defined as the solution of the equation with a delta source. Therefore, for Klein-Gordon equation, we have

$$\left(L_\mu L^\mu + m_0^2\right)G\left(\theta - \tilde{\theta}\right) = -\delta^4\left(\theta - \tilde{\theta}\right). \tag{6.3}$$

If $G\left(\theta - \tilde{\theta}\right)$ is known, then we can write the solution of Eqn. (6.1) as

$$\phi(\theta) = -\int d^4\tilde{\theta} G\left(\theta - \tilde{\theta}\right)J\left(\tilde{\theta}\right) \tag{6.4}$$

then

$$\left(L_\mu L^\mu + m_0^2\right)\phi(\theta) = -\int d^4\tilde{\theta}\left(L_{\theta\mu} L_\theta^\mu + m_0^2\right)G\left(\theta - \tilde{\theta}\right)J\left(\tilde{\theta}\right) \tag{6.5}$$

$$= -\int d^4\tilde{\theta}\left(-\delta^4\left(\theta - \tilde{\theta}\right)\right)J\left(\tilde{\theta}\right) \tag{6.6}$$

$$= J(\theta).$$

(6.7)

If we Fourier transform the functions, it would turn the above partial differential equation into an algebraic equation. Therefore, define

$$\delta^4(\theta-\tilde{\theta}) = \frac{1}{(2\pi)^4}\int d^4 J e^{-iJ(\theta-\tilde{\theta})} \tag{6.8}$$

and

$$G(\theta-\tilde{\theta}) = \frac{d^4 J}{(2\pi)^4}\int e^{-iJ(\theta-\tilde{\theta})}\widehat{G}(J). \tag{6.9}$$

Substituting above values in Eqn. (6.3), we get

$$\widehat{G}(J) = \frac{1}{J^2 - m_0^2}. \tag{6.10}$$

Thus

$$G(\theta-\tilde{\theta}) = \frac{d^4 J}{(2\pi)^4}\int \frac{e^{-iJ(\theta-\tilde{\theta})}}{J^2 - m_0^2}. \tag{6.11}$$

Note that the above Green's function has poles at $J^0 = \pm E_J$. This poles can be removed using advanced Green's function. We would not go into details as it is not the aim of our paper, although, readers can refer to [8] (pg. 194) to make themselves familiarise with the advance Green's function corresponding to the Cartesian Klein-Gordon equation

VII. COMPLEX KLEIN-GORDON FIELD EQUATION

In complex Klein-Gordon field theory on $R \times S^3$ topology, we have

$$(L_\mu L^\mu + m_0^2)\phi(\theta) = 0 \tag{7.1}$$

$$(L_\mu L^\mu + m_0^2)\phi^\dagger(\theta) = 0 \tag{7.2}$$

where $\phi(\theta) \neq \phi^\dagger(\theta)$. We can express $\phi(\theta)$ and $\phi^\dagger(\theta)$ in terms of two distinct spin zero scalar fields $\phi_1(\theta)$ and $\phi_2(\theta)$ which are hermitian. Thus

$$\phi(\theta) = \frac{1}{\sqrt{2}}\big(\phi_1(\theta) + i\phi_2(\theta)\big) \tag{7.3}$$

$$\phi^\dagger(\theta) = \frac{1}{\sqrt{2}}\big(\phi_1(\theta) - i\phi_2(\theta)\big). \tag{7.4}$$

Inverting the above relations yields

$$\phi_1(\theta) = \frac{1}{\sqrt{2}}\big(\phi(\theta) + \phi^\dagger(\theta)\big) \tag{7.5}$$

$$\phi_2(\theta) = \frac{-i}{\sqrt{2}}\big(\phi(\theta) - \phi^\dagger(\theta)\big). \tag{7.6}$$

We can now express Eqn. (7.1) and (7.2) in terms of $\phi_1(\theta)$ and $\phi_2(\theta)$ to get

$$(L_\mu L^\mu + m_0^2)\phi_1(\theta) = 0 \tag{7.7}$$

$$(L_\mu L^\mu + m_0^2)\phi_2(\theta) = 0. \tag{7.8}$$

Corresponding Lagrangian from which we can derive the above equations of motion is

$$\mathcal{L} = \frac{1}{2}L_\mu\phi_1 L^\mu\phi_1 + \frac{1}{2}L_\mu\phi_2 L^\mu\phi_2 - \frac{m_0^2}{2}(\phi_1^2 + \phi_2^2) \quad (7.9)$$

$$= \frac{1}{2}L_\mu\phi^\dagger L^\mu\phi - m_0^2\phi^\dagger\phi. \quad (7.10)$$

If we write solutions in terms of $\phi_1(\theta)$ and $\phi_2(\theta)$ the we can define the conjugate momenta as

$$\Pi_i(\theta) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_i(\theta)} = \dot{\phi}_i(\theta) \quad (7.11)$$

where $- = 1,2$. Commutation relations can be given by

$$\{\phi_i(\theta), \phi_j(\tilde{\theta})\}_{\theta^0=\tilde{\theta}^0} = \{\Pi_i(\theta), \Pi_j(\tilde{\theta})\}_{\theta^0=\tilde{\theta}^0} = 0 \quad (7.12)$$

$$\{\phi_i(\theta), \Pi_j(\tilde{\theta})\}_{\theta^0=\tilde{\theta}^0} = i\delta_{ij}\delta^3(\theta - \tilde{\theta}). \quad (7.13)$$

And finally, the Hamiltonian can be given by

$$\mathcal{H} = \sum_i (\Pi_i\phi_i) - \mathcal{L} \quad (7.14)$$

$$= \sum_i \left(\frac{1}{2}\Pi_i^2 + \frac{1}{2}L_\mu\phi_i . L_\mu\phi_i + \frac{m_0^2}{2}\phi_i^2\right). \quad (7.15)$$

Now, on the other hand, if we write our solution in terms of ϕ and $\phi^\dagger$, we can define conjugate momenta as follow:

$$\Pi(\theta) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^\dagger(\theta)} = \dot{\phi}(\theta) = \frac{1}{\sqrt{2}}\left(\dot{\phi}_1(\theta) + i\dot{\phi}_2(\theta)\right) = \frac{1}{\sqrt{2}}(\Pi_1(\theta) + i\Pi_2(\theta)) \quad (7.16)$$

$$\Pi^\dagger(\theta) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\theta)} = \dot{\phi}^\dagger(\theta) = \frac{1}{\sqrt{2}}\left(\dot{\phi}_1(\theta) - i\dot{\phi}_2(\theta)\right) = \frac{1}{\sqrt{2}}(\Pi_1(\theta) - i\Pi_2(\theta)). \quad (7.17)$$

Equal time commutation relations are given by

$$[\phi(\theta), \phi(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = [\phi(\theta), \phi^\dagger(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = [\phi^\dagger(\theta), \phi^\dagger(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = 0 \quad (7.18)$$

$$[\Pi(\theta), \Pi(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = [\Pi(\theta), \Pi^\dagger(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = [\Pi^\dagger(\theta), \Pi^\dagger(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = 0 \quad (7.19)$$

$$[\phi(\theta), \Pi^\dagger(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = [\phi^\dagger(\theta), \Pi(\tilde{\theta})]_{\theta^0=\tilde{\theta}^0} = i\delta^3(\theta - \tilde{\theta}). \quad (7.20)$$

Using (7.10), we can write the Hamiltonian density as

$$\mathcal{H} = \Pi\dot{\phi}^\dagger + \Pi^\dagger\dot{\phi} - \mathcal{L} \quad (7.21)$$

$$= \Pi^\dagger\Pi + L_\mu\phi^\dagger . L_\mu\phi + m_0^2\phi^\dagger\phi \quad (7.22)$$

and thus

$$H = \int d^3\theta\left(\Pi^\dagger\Pi + L_\mu\phi^\dagger . L_\mu\phi + m_0^2\phi^\dagger\phi\right) \quad (7.23)$$

Solutions of Eqn. (7.1) and (7.2) can be given as

$$\phi_i(\theta) = \int \frac{d^3J}{\sqrt{(2\pi)^3 2J^0}}\left(e^{-iJ.\theta}a_i(J) + e^{iJ.\theta}a_i^\dagger(J)\right). \tag{7.24}$$

Similarly, the solutions of Eqn. (7.1) and (7.2) can be given by

$$\phi(\theta) = \int \frac{d^3J}{\sqrt{(2\pi)^3 2J^0}}\left(e^{-iJ.\theta}a(J) + e^{iJ.\theta}b^\dagger(J)\right) \tag{7.25}$$

$$\phi^\dagger(\theta) = \int \frac{d^3J}{\sqrt{(2\pi)^3 2J^0}}\left(e^{-iJ.\theta}b(J) + e^{iJ.\theta}a^\dagger(J)\right) \tag{7.26}$$

where

$$a(J) = \frac{1}{\sqrt{2}}\left(a_1(J) + ia_2(J)\right) \tag{7.27}$$

$$b(J) = \frac{1}{\sqrt{2}}\left(a_1(J) - ia_2(J)\right) \tag{7.28}$$

Commutation relations for annihilation and creation operators can be given as

$$\left[a_i(J), a_j^\dagger(J')\right] = \delta_{ij}\delta^3(J - J') \tag{7.29}$$

$$\left[a(J), a^\dagger(J')\right] = \left[b(J), b^\dagger(J')\right] = \delta^3(J - J'). \tag{7.30}$$

All other commutation relations are zero.

VIII. CONCLUSION

In this paper, we have established $R \times S^3$ Klein-Gordon field theory for both complex and scalar fields. Furthermore, corresponding Hamiltonin and commutation relations within operators are derived. This approach to Klein-Gordon field theory will be important in the problems with angular dependence instead of Mikowskian distance. An example of such problem with angular dependence can be found in the beginning of [7].

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Cite this article as :

Omprakash Atale, "Quantization of RxS3 Topological Klein-Gordon Field Theory", International Journal of Scientific Research in Science and Technology (IJSRST), Online ISSN : 2395-602X, Print ISSN : 2395-6011, Volume 9 Issue 1, pp. 206-217, January-February 2022. Available at doi : https://doi.org/10.32628/IJSRST229136

Journal URL : https://ijsrst.com/IJSRST229136