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Study On Uniaxial and Biaxial Crystals in Finsler Space

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ABSTRACT

In this paper we have studied the application of Finsler geometry to physics is crystal optics and the symmetry properties. The transparent crystals fall into only three distinct classes from the point of view of their optical properties, biaxial, uniaxial and isotropic crystals. Here we have studied biaxial crystal and uniaxial crystal, since the isotropic crystals behave optically as amorphous bodies they have no optical anisotropy and correspond to Euclidean geometry.

Keywords : Crystal Optics, Uniaxial, Biaxial, Optical Anisotropy.

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I. INTRODUCTION

Many researchers Born, M. [1], Born, M. and Wolf [2], P. L. Antonelli, R. S. Ingarden and M. Matsumoto[3] and *C. W.* Bunn [4] are studied on Crystal Optics in Finsler space. Boguslavsky, G. Yu [5] studied theory of Locally Anisotropic Space-Time. Born and Wolf [2], studied the Optical theory is based on two way: Maxwell's equations and Material equations.

The material equations in an isotropic medium are given by

(1.1) (a) $\boldsymbol{j} = \sigma \boldsymbol{E}$, (b) $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, , (c) $\boldsymbol{B} = \mu \boldsymbol{H}$,

here σ is specific conductivity, ε is dielectric constant and μ is magnetic permeability.

In dealing with crystals we have generalized these later equations in the view of anisotropy. We consider that the medium is homogeneous, non-conducting, and magnetically isotropic, there are also magnetic crystals, but as the effect of magnetization on optical phenomena is small, the magnetic anisotropy may be neglected (Boguslavsky, G. Yu [5]). We consider substances whose electrical excitations depend on the direction of the electric field. The equation (1.1b) we assume the relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ to have the simplest form which can account for anisotropic behavior, which each component of $\boldsymbol{D}$ is linearly related to the components of $\mathbf{E}$, we can written as

$$(1.2) \quad D_k = \sum_l \varepsilon_{kl} E_l$$

where k stands for one of the three indices x, y, and z, and l stands for each of x, y and z in turn in the summation.

The most direct and simple application of Finsler geometry to physics is crystal optics. In crystals the electric vector $\boldsymbol{E} = E_i$, $i = 1,2,3$ is not in general parallel to the electric displacement vector $\boldsymbol{D} = D_i$, the set of equation (1.2) can be written as (*C. W.* Bunn [4])

$$(1.3) \quad D_i = \varepsilon_{ij} E_j$$

where ε_{ij} is the dielectric tensor , here we use Einstein's summation convention, but we do not distinguish contravariant and covariant indices since coordinates are orthogonal Cartesian.

Here we have studied two classes: biaxial and uniaxial, since the isotropic crystals behave optically as amorphous bodies they have no optical anisotropy and correspond to Euclidean geometry *(C. W.* Bunn [4]). The dielectric axes are those corresponding to the eigen vectors and eigen values of tensor ε_{ij} in (1.3) which is assumed to be real and symmetric.

(1.4) $D_i = \varepsilon_i E_i$, $v_i = \frac{c}{\sqrt{\mu \varepsilon_i}}$, $(i = 1, 2, 3)$.

where v_i is velocities of propagation of light in the crystal (R. Courant[7]). The ellipsoid of wave normal is formed the expression for electric energy density in the coordinate system of the dielectric axes (Born, M. and Wolf [2]),

(1.5) $C^2 = 8\pi\omega_e = \boldsymbol{E}.\boldsymbol{D} = \frac{{D_1}^2}{\varepsilon_1} + \frac{{D_2}^2}{\varepsilon_2} + \frac{{D_3}^2}{\varepsilon_3}$,

If we take $x = \frac{D_1}{C}, y = \frac{D_2}{C}, z = \frac{D_2}{C}$ then we can write

(1.6) $\frac{x^2}{\varepsilon_1} + \frac{y^2}{\varepsilon_2} + \frac{z^2}{\varepsilon_3} = 1$

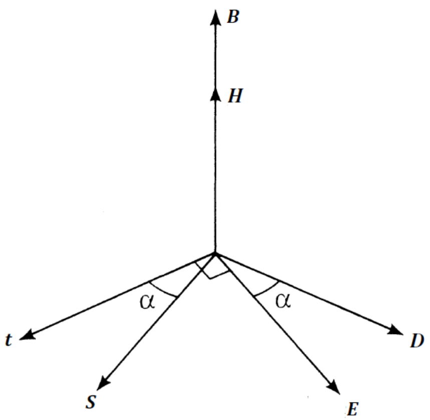

Fig. 1. Directions of the wave normal of field vectors and of the energy flow in an anisotropic crystals.

Here, we construct the directions of vibrations of the $\boldsymbol{D}$ vectors belonging to a wave normal $\boldsymbol{s}$ as axes of the section through the origin normal to **s** and the directions of optical axes of the crystal as normals to spherical sections of the ellipsoid through the origin. Since ellipsoid can be triaxial, biaxial or uniaxial, and the optical axes can be one or two.

The distinguish between phase velocity v_p in direction $\boldsymbol{s}$ and the ray velocity v_r which is the velocity of energy transport in direction $\boldsymbol{t}$, (Born, M. and Wolf [2]) we have

(1.7) $v_p = \frac{c}{n} = v_r\, \boldsymbol{t}.\boldsymbol{s} = v_r \cos\alpha = \frac{|S|}{\omega}\cos\alpha$, $\omega = \omega_e + \omega_m = 2\omega_e$

where n is the refraction index, and Fig.1, it can be shown that as a condition of solvability of the problem one obtains the Fresnel equation of wave normals ${s_1}^2 + {s_1}^2 + {s_1}^2 = 1$, $\boldsymbol{s} = s_i$ (Born, M. and Wolf [2]),

(1.8) ${s_1}^2({v_p}^2 - {v_2}^2)({v_p}^2 - {v_3}^2) + {s_2}^2({v_p}^2 - {v_3}^2)({v_p}^2 - {v_1}^2) + {s_3}^2({v_p}^2 - {v_1}^2)({v_p}^2 - {v_2}^2) = 0$.

Now we have studies the case of an optically uniaxial crystal. Let us assume that the optical axis is in the three directions, $v_1 = v_2 = v_0$ for ordinary velocity and $v_3 = v_e$ for extraordinary velocity. Then the set of equation (1.8) can be written as

(1.9) $({v_p}^2 - {v_0}^2)({v_p}^2 - {v_e}^2)\sin^2\theta + ({v_p}^2 - {v_0}^2)\cos^2\theta = 0$,

where (1.10) ${s_1}^2 + {s_2}^2 = \sin^2\theta$, ${s_3}^2 = \cos^2\theta$,

(1.11) ${v'_p}^2 = {v_0}^2$,

(1.12) ${v''_p}^2 = {v_0}^2 \cos^2\theta + {v_e}^2 \sin^2\theta$

II. UNIAXIAL CRYSTALS

We have using the method Okubo [6], the equation of the extraordinary normal surface (1.12) in orthogonal Cartesian coordinates. And we have written the well known relations between spherical and rectilinear orthogonal coordinates (Born and Wolf [2])

(1.13) (a) $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$, $z = r\cos\theta$

(b) $r^2 = x^2 + y^2 + z^2$, $\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$ and $\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$.

Putting these relations in the equation (1.12), we have ($v_0 = a$, $v_e = b$)

(1.14) $x^2+y^2+z^2=a^2\frac{z^2}{x^2+y^2+z^2}+b^2\frac{x^2+y^2}{x^2+y^2+z^2}$, $a\neq b,\ a,b\neq 0$.

Multiplying by $(x^2+y^2+z^2)$ in (1.14), we get

(1.15) $(x^2+y^2+z^2)^2-a^2z^2-b^2(x^2+y^2)=0,\quad x,y,z\neq 0$.

Using the method Okubo [6], and we write

(1.16) $1^i=\frac{y^i}{L(x,y)}$, $i=1,2,3$,

substitute this for x,y,z in (1.15) and solving for L, we get

(1.17) $L(x,y)=\frac{(y^1)^2+(y^2)^2+(y^3)^2}{\sqrt{a^2(y^3)^2+b^2[(y^1)^2+(y^2)^2]}}$,

$y\neq 0,\quad a,b\neq 0$

Equation (1.17) is written in special coordinates adjusted to the symmetry of our example. The Lagrangian of the type

(1.18) $L(x,y)=\frac{a_{ij}(x)y^iy^j}{\sqrt{b_{ij}(x)y^iy^j}}$, $\quad a_{ij}(x)\neq b_{ij}(x)$,

Here both quadratic forms $\alpha^2=a_{ij}(x)y^iy^j$ and $\beta^2=b_{ij}(x)y^iy^j$ are different and positive definite.

III. BIAXIAL CRYSTALS

The Fresnel equation (1.18) with respect to the coordinates of 6th order equation, as we have seen above, but this time it cannot be factorized into a 2nd order and a 4th order equations, although it is possible in each normal section of the surface by the coordinate planes $x=0$, $y=0$ and $z=0$. To uniquely fix the appearance of the intersection curves, let us label the three axes so that

(1.19) (a) $\varepsilon_x<\varepsilon_y<\varepsilon_z(\varepsilon_x=\varepsilon_1,\ \varepsilon_y=\varepsilon_2,\varepsilon_z=\varepsilon_3)$

or (b) $v_x>v_y>v_z\ (v_x=v_1,v_y=v_2,v_z=v_3)$

It is easily seen that, (Born and Wolf [2]), the sections of our surface by the three coordinate planes give forms present on Fig.2. and these forms can be combined in the 3-dimensional picture shown in Fig.3. We obtained a 2-sheet surface whose two sheets contact only in 4 points corresponding to the point N_1 in the positive quadrant *x, y, z.* These are intersection points of the normal surface by the two optical axes lying in the plane *(x, z)* under the angle β with the z-axis. All the normal surface is smooth everywhere, but if we divide it into the outer sheet and the inner sheet, both sheets have points of non-differentiability in points of contact N_i *(i* = 1,2,3,4), where cusps occur.

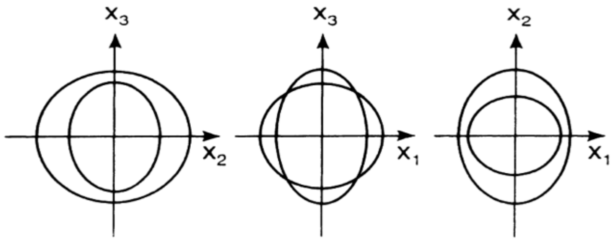

Fig. 2.: Sections of the normal surface of a biaxial crystal by three coordinates planes through the principal dielectric axes.

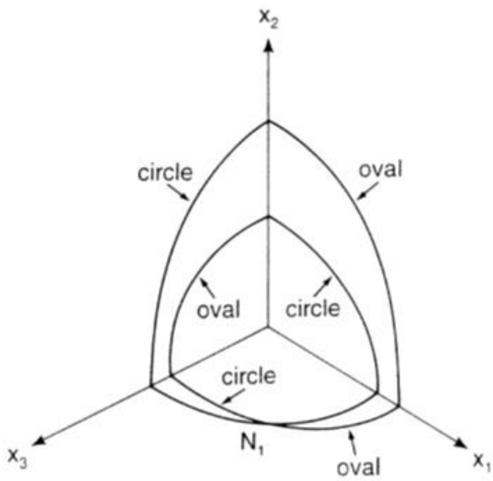

Fig. 3. : The normal surface of a biaxial crystal

The Fresnel equation (1.18) is easily soluble if we introduce the difference variables q_x , q_z ,q ,

(1.20) $v_x^2=v_y^2+q_x$, $\quad v_z^2=v_y^2+q_z$, $\quad v_p^2=v_y^2+q$.

Then the Fresnel equation can be written as

(1.21) $q^2+[{s_x}^2q_z+{s_y}^2(q_z-q_x)-{s_z}^2q_z]q-{s_y}^2q_xq_z=0$

and its solution as

(1.22) $q=-\frac{1}{2}P\pm\frac{1}{2}\sqrt{\Delta}$,

where

(1.23) $P=q_z-q_x-(q_x+q_z)cos\theta_1cos\theta_2$,

(1.24) $Q=[(q_z+q_x)sin\theta_1sin\theta_2]^2$,

(1.25) $cos\theta_1=s_x\,sin\beta+s_zcos\beta$,

(1.26) $cos\theta_2=-s_x\,sin\beta+s_zcos\beta$, (1.27) $tan\beta = \pm\sqrt{\frac{q_x}{q_z}} = \pm\sqrt{\frac{v_x^2-v_y^2}{v_y^2-v_z^2}}$

Finally, we can express the solutions (there are two) of the Fresnel equation as

(1.28) $v_p^2 = \frac{1}{2}[v_x^2 + v_z^2 + (v_x^2 - v_z^2)cos(\theta_1 \pm \theta_2)]$

or, introducing the new constants

(1.29) $A^2 = \frac{1}{2}(v_x^2 + v_z^2) > 0, \quad B^2 = \frac{1}{2}(v_x^2 - v_z^2) > 0$, $A^2 > B^2$

(1.30) $v_p^2 = A^2 + B^2 cos(\theta_1 \pm \theta_2)$.

The dependence on the third constant v_y is implicitly contained in the angles β (1.27) and θ_1, θ_2 (1.25)-(1.26). In order to get a more explicit representation of the result let us introduce *a* (in general, skew) coordinate system composed of the y-axis and the two optical axes,

(1.31) $x = (\xi - \zeta)sin\beta\,, \quad y = \eta\,, \quad z = (\xi + \zeta)cos\beta\,,$

The determinant of the transformation (1.31)

(1.32) $D = \begin{bmatrix} sin\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ cos\beta & 0 & cos\beta \end{bmatrix} = 2sin\beta cos\beta = sin2\beta \neq 0, \;\; 0 < \beta < \frac{1}{2}\pi$.

According to the assumption that the crystal is optically biaxial for uniaxial crystals the transformation (1.31) makes no sense.

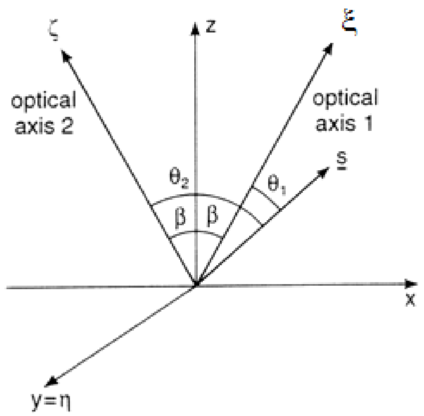

Fig.4. : Dielectrical coordinate system (x, y, z) and optical coordinate system (ξ, η, ζ) in a biaxial crystal.

Introducing the abbreviation,

(1.33) $C = cos^2\beta - sin^2\beta\,, \quad -1 < C < +1$

Containing constant v_y , (1.27), we may write (1.30) in the form

(1.34) $\xi^2 + \eta^2 + \zeta^2 + 2C\xi\zeta = A^2 + B^2 \frac{(\xi - C\zeta)(\zeta + C\xi) \mp [\eta^2 + \zeta^2(1-C^2) + 4C\xi\zeta]^{\frac{1}{2}} [\xi^2(1-C^2) + \eta^2]^{\frac{1}{2}}}{\xi^2 + \eta^2 + \zeta^2 + 2C\xi\zeta}$

Using the method Okubo [6] substitution as in (1.16) we finally obtain two metric functions,

$Q^2 = {y^1}^2 + {y^2}^2 + {y^3}^2 + 2Cy^1y^3$

And M $= [{y^2}^2 + {y^3}^2(1 - C^2) + 4Cy^1y^3]^{\frac{1}{2}}[{y^1}^2(1 - C^2) + {y^2}^2]^{\frac{1}{2}}$

is conveniently given by

(1.35) $L\pm = \frac{Q^2}{\sqrt{A^2Q^2 + B^2\{(y^1 - Cy^3)(y^3 + Cy^1) \mp M}}$

If the optical axes are orthogonal, $\beta = \frac{\pi}{4}$, $C = 0$, (1.35) considerably simplifies to

(1.36) $L\pm = \frac{{y^1}^2 + {y^2}^2 + {y^3}^2}{\sqrt{A^2[{y^1}^2 + {y^2}^2 + {y^3}^2] + B^2\{(y^1y^3 \mp ({y^2}^2 + {y^3}^2)^{\frac{1}{2}}({y^1}^2 + {y^2}^2)^{\frac{1}{2}}]}}$

We see that under conditions (1.29) and (1.33) about constants functions $L\pm$ are always positive definite, convex, (1) p-homogeneous in y, and smooth in the domain $D = R^3 - S$,

(1.37) $S = \{1) y^1 = y^2 = y^3 = 0, \quad 2) \; y^1 \neq 0, \; y^2 = y^3 = 0, \;\; 3) \; y^1 = y^2 = 0, y^3 \neq 0\}$,

So they define two regular conjugated Minkowski spaces two regular conjugated Finsler spaces for fluid crystals when constants A, B, C can depend on the position x. The condition of smoothness everywhere is not important physically, but the exceptions from smoothness represent interesting mathematical and physical facts. In the mathematical view both indicatricies can be considered as a one whole and we can also speak about multi-Finsler spaces having indicatricies with many sheets connected by one algebraic equation. These spaces have many sets of trajectories and other properties, which however are mutually connected. As we know physics requires such entities, although they can be also separated for instance, by means of selection of polarization using Nicol prisms etc. In the case of uniaxial crystals the situation is simpler, but we have actually two indicatricies (i.e. sphere and oval), but they contact smoothly at the optical axis. In the previous section we neglected the sphere as giving an Euclidean space, but actually the sphere belongs to the complete physical phenomenon.

IV. CONCLUSION

In this paper we have studied biaxial crystal and uniaxial crystal, since the isotropic crystals behave optically as amorphous bodies they have no optical anisotropy and correspond to Euclidean geometry. In this continuation we have studied the case of an optically uniaxial crystal and assume that the optical axis is in the three directions, $v_1 = v_2 = v_0$ for ordinary velocity and $v_3 = v_e$ for extraordinary velocity and obtained some results. In uniaxial crystal we have studied using the method Okubo, the equation of the extraordinary normal surface (1.12) in orthogonal Cartesian coordinates and using the well known relations between spherical and rectilinear orthogonal coordinates, and obtained the some results. Which are known as Fresnel equation in the type of Lagrangian L. In the biaxial crystal we have seen that, the sections of our surface by the three coordinate planes give forms present on Fig.2. and these forms can be combined in the 3-dimensional picture shown in Fig.3. We obtained a 2-sheet surface whose two sheets contact only in 4 points corresponding to the point N_1 in the positive quadrant x, y, z. These are intersection points of the normal surface by the two optical axes lying in the plane (x, z) under the angle β with the z-axis. All the normal surface is smooth everywhere, but if we divide it into the outer sheet and the inner sheet, both sheets have points of non-differentiability in points of contact N_i $(i = 1,2,3,4)$, where cusps occur. We see that under conditions (1.29) and (1.33) about constants functions $L \pm$ are always positive definite, convex, (1) p-homogeneous in y, and smooth in the domain $D = R^3 - S$, so they define two regular conjugated Minkowski spaces two regular conjugated Finsler spaces. The condition of smoothness everywhere is not important physically, but the exceptions from smoothness represent interesting mathematical and physical facts. In the mathematical view both indicatricies can be considered as a one whole, and we can also speak about multi-Finsler spaces having indicatricies with many sheets connected by one algebraic equation. These spaces have many sets of trajectories and other properties, which however are mutually connected. In the case of uniaxial crystals the situation is simpler, but we have actually two indicatricies (i.e. sphere and oval), but they contact smoothly at the optical axis. Spaces (1.35) and (1.36) have not been yet investigated mathematically, as far as we know. We think that it may be interesting to investigate further mathematical properties of these spaces, in particular, their torsion for the Minkowski case and curvature for the general Finsler case.

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