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Study of Finite Dimensional Topological Vector Spaces

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ABSTRACT

In this present paper, we studied about finite dimensional topological vector spaces. In general analysis it is customary to study linear spaces for which there is defined a "norm," which takes the place of the absolute value of ordinary analysis in defining distance, limit point, continuity, and so on. Linear metric spaces more general than the normed spaces have also been studied by Fréchet, Banach, and several others.

Keywords: TVS, Finite Dimensional Spaces, Linear Topology Spaces, Hausdorff Space.

I. INTRODUCTION

In line with the trend toward general topology, it seems natural to generalize still more by introducing linear topological spaces, that is, linear spaces which are at the same time topological spaces, in which the fundamental "linear*' operations of addition and scalar multiplication are continuous. We shall always assume that the topology is subject to the axioms for a T1-space [1-2].

The topology of T1-spaces may be introduced in various ways; by postulating a system of open sets or of neighborhoods with certain properties, and so on. We shall find it convenient to give a set of postulates for the topology of the linear space L in which "neighborhood" is the fundamental undefined notion. Since L is a topological group, it has a uniform topology and hence it is sufficient to consider neighborhoods of the origin. Moreover, the "uniform structure" implies that L is a completely regular Hausdorff space [3].

II. FINITE-DIMENSIONAL SPACES

Finite dimensional spaces occur often in this business. It is nice to know, and should be no surprise, that they are the same familiar objects that we are used to.

Proposition. Any Hausdorff topology on a finite dimensional vector space with respect to which vector space operations are continuous is equivalent to the usual one.
It is not necessary to assume the topology to be locally convex.

Proof. A basis of V determines a linear isomorphism $f : \mathbb{R}^n \rightarrow V$, which is continuous by assumption on the topology of V. By definition of continuity, if U is any neighbourhood of 0 in V there exists some disk $B(r)$ with $f\,B(r) \subseteq U$.

It remains to show that the inverse of f is continuous. For this, it suffices to show that f neighbourhood of 0 [4].

Since the topology of V is Hausdorff, for every point s in $f(\mathbb{S}^{n-1})$ By Lemma 2.1 we may assume U to be balanced. Since U is balanced, the neighbourhood U/2 is contained in U, and does not meet s+U/2. Since the embedding of $\mathbb{S}^{n-1}$ is continuous, we may find a neibhourhood $\sum_s$ of s in $\mathbb{S}^{n-1}$ such that $f(\Sigma_s)$ is contained in s+U/2, and which does not intersect U/2. Since $\mathbb{S}^{n-1}$ is compact, we may find a finite number of Σ_s covering $\mathbb{S}^{n-1}$. The intersection of this finite collection of sets U_s is still a neigbhourhood of 0, and does not intersect $\mathbb{S}^{n-1}$. Since it is balanced, it does not contain any point exterior to $\mathbb{S}^{n-1}$, either. So it is contained in the open disk $\|v\| < 1$.

Proposition. Any finite dimensional subspace E of a TVS is closed.

I am reverting here to the convention that a TVS is locally convex, although it is not a necessary assumption.

Proof. It must be shown that the complement of E in the TVS V is open, or that every point of V has a neighbourhood containing no point of E. The previous Lemma tells us that we may find a neighbourhood of $\overline{0}$ in V whose intersection with E is contained inside a unit sphere. There therefore exists a semi norm ρ of V defining the topology of E. Since

$$\|e\|\rho \leq \|e - v\|\rho + \|v\|\rho, \|e - v\|\rho \geq \|e\|\rho - \|v\|\rho$$

Choose $R = \|v\|_\rho$. Then for $\|e\|_\rho > 2R$

$\|e - v\|_\rho \geq \|e\|_\rho - \|v\|_\rho > 2R - R = R$

Therefore the minimum value m of $\|e - v\|_\rho$ on the compact disk $\|e\|_\rho \leq 2R$, which is at most $\|0 - v\|_\rho = R$, is the minimum value on all of E. The disk $\|x - v\|_\rho < m$ is then a neighbourhood of v containing no point of E.

Corollary. If U is a closed linear subspace of a Hausdorff TVS and F a finite dimensional subspace, then $U + F$ is closed [5].

Proof. Let V be the TVS. The claim is true because the image of F in V/U is closed.

Proposition. Suppose U to be of finite codimension in the TVS V. Then U is closed if and only if every linear function on V vanishing on U is continuous.

Proof. Let $E = V/U$. According to Proposition 2.8 and Proposition 3.1, U is closed if and only if the quotient topology on E is the usual one. But a finite dimensional TVS is Hausdorff if and only if every linear functionon it is continuous [6-7].

As a special case:

Corollary. If f is a linear function on V, it is continuous if and only if the hyperplane $f = 0$ is closed.

The following is used in the theory of partial differential equations, among other places, to verify that an eigenspace has finite dimension, and in the theory of complex analytic manifolds to verify that certain cohomology groups have finite dimension.

Proposition. Any locally compact Hausdorff TVS is finite dimensional.

Proof. Let Ω be a compact neighbourhood of 0. Given any $0 < c < 1$ the set $\overline{\Omega}$ may be covered by a finite number of $e_i + c\,\Omega$. Let E be the space spanned by the e_i, which is closed in $\overline{V}$. Let $V = V/E$. The image of Ω in $\overline{V}$ is contained in $c\,\Omega$, which implies that each $c^{-n}\Omega$ is contained in Ω. This implies that Ω = V. But

{0} is the only compact Hausdorff TVS.

III. REFERENCES

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