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Study of Modern Methods in Topological Vector Spaces

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ABSTRACT

In this present paper, we studied about modern methods in topological vector spaces. A topological vector space is one of the basic structures investigated in functional analysis. The elements of topological vector spaces are typically functions or linear operators acting on topological vector spaces, and the topology is often defined so as to capture a particular notion of convergence of sequence of functions. Hilbert and Banach spaces are well known examples unless stated otherwise, the underlying field of a topological vector space is assumed to be either the complex number 'C ' or the real number 'R' [1-2].
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I. INTRODUCTION

In topology and related areas of mathematics a topological property or topological invariant is a property of a topological space which is invariant under homomorphism. That is, a property if whenever a space X possesses that property every space homomorphic to X possesses that property. Informally, a topological property is a property of space that can be expressed using open sets. A common problem in topology is to decide whether two topological spaces are homomorphic or not. To prove that two spaces are not homomorphic, it is sufficient to find a topological property which is not shared by them [3].

A vector space is an abelian group with respect to the operation of addition and in and in a topological vector space. The inverse operation is always continuous (since it is same as multiplication by -1). Hence every topological vector space is an abelian topological group [7].

In Functional Analysis and related areas of mathematics, locally convex topological vector spaces or locally convex spaces are examples of topological vector spaces (TVS) that generalize normed spaces. They can be defined as topological vector spaces whose topology is generated by translations of balanced, absorbent, convex sets. Alternatively, they can be defined as a vector space with a family of semi norms, and a topology can be defined in terms of that family. Although in general such spaces are not necessarily norm able, the existence of a convex local base for the zero vector is strong enough for the Hahn - Banach theorem to hold, yielding a sufficiently rich theory of continuous linear functionals [4].

II. MODERN METHODS IN TOPOLOGYCAL VECTOR SPACES

A Hilbert space V is a complex vector space assigned a positive definite inner product $u \bullet v$ with the property that Cauchy sequences converge. These are conceptually the simplest topological vector spaces, with the topology defined by the condition that a subset U of V is open if and if it contains a neighbourhood $||v-u|| < \varepsilon$ for every one of the points u in U. But there are other naturally occurring spaces in which things are a bit more complicated. For example, how do you measure how close two functions in $C^\infty(\mathbb{R}/\mathbb{Z})$ are? Under what circumstances does a sequence of functions f_n in $C^\infty(\mathbb{R}/\mathbb{Z})$ converge to a function in that space? If two functions in $C^\infty(\mathbb{R}/\mathbb{Z})$ are close then their values should be close, but you should also require that their derivatives be close [5]. So, you introduce naturally an infinite number of measures of difference:

$$||f||_m = \sup_x \left|f^{(m)}(x)\right|,$$

and say that $f_n \to f$ if $||f_n - f||_m \to 0$ for all m. The most fruitful way to put topologies on many other infinite-dimensional vector spaces is by using measures of the size of a vector that are weaker than those on Hilbert spaces.

Proposition. If ρ is a non-negative function on the vector space V, the following are equivalent:

(a) *for all scalars a and b,* $\rho(au+bv) \leq |a|\rho(u) + |b|\rho(v)$;

(b) *for any scalar* a, $\rho(av)|a|\rho(v)$, and $\rho(tu+(1-t)v) \leq t\rho(u) + (1-t)\rho(v) for\ all\ t\ in$ [0,1];

(c) *for any scalar a,* $\rho(av)=|a|\rho(v)$, and $\rho(u+v) \leq \rho(u) + \rho(v)$

The real-valued function ρ on a vector space is said to be convex if,

$\rho(tu + (1-t)v) \leq t\rho(u) + (1-t)\rho(v)$

for all $0 \leq t \leq 1$, and these conditions are essentially variations on convexity.

Any function ρ satisfying these conditions will be called a semi-norm. It is called a norm if $\rho(v) = 0$ implies $v = 0$.

A prototypical norm is the function $||x||\sqrt{|x_1|^2 + \cdots + |x_n|^2} in\ \mathbb{C}^n$ or the integral

$$\int_{\mathbb{R}^n} |f(x)|^2 dx_1 \ldots dx_n$$

for $f$in the space of continuous functions on R^n of compact support. The functions $\|f\|_m$ on $C^\infty(\mathbb{R}/\mathbb{Z})$ are semi norms[6].

Proof. That (c) implies (b) and that (b) implies (a) is immediate. Assuming (a), we have

$\rho(v) = \rho(a^{-1}av) \leq |a|^{-1}\rho(av) \leq |a|^{-1}\,|a|\rho(v) = \rho(v)$,

leading to homegeity.

Corollary. The kernel

ker $(\rho)=\{v\in V/\ \rho(v)=0\}$

of a semi-norm on a vector space is a linear subspace.

From now on I'll usually express semi norms in norm notation—$\|v\|_\rho$ instead of $\rho(v)$.

The conditions on a semi norm can be formulated geometrically, and in two rather different ways. The graph of a semi norm ρ is the set Γ_ρ of pairs $(v, \|v\|_\rho)$ in $V \oplus \mathbb{R}$. Let Γ_ρ^+ be the set of pairs (v, r) with $r > \|v\|_\rho$. The conditions for ρ to be a semi norm are that Γ_ρ^+ be convex, stable under rotations $(v, r) \mapsto (cv, r)$ for $|c| = 1$, and homogeneous with respect to multiplication by positive scalars. (I recall that a subset of a vector space is convex if the real line segment connecting two points in it is also in it.)

A more interesting geometric characterization of a semi norm is in terms of the disks associated to it. If ρ is a semi norm its open and closed disks are defined as

$$B_\rho(r-) = \{v \,|\, \|v\|_\rho < r\}$$

$$B_\rho(r) = \{v \,|\, \|v\|_\rho \le r\}$$

We shall see in a moment how semi norms can be completely characterized by the subsets of V that are theirunit disks. What are the necessary conditions for a subset of a vector space to be the unit disk of a semi norm?

A subset of V is **balanced** if cu is in it whenever u is in it and $|c| = 1$, and **strongly balanced** if cu is in it whenever u is in it and $|c| \le 1$. (This is not standard terminology, but as often in this business there is no standard terminology.) Convex and balanced implies strongly balanced [7].

A subset X of V is **absorbing** if for each v in V there exists ε such that $cv \in X$ for all $|c| < \varepsilon$. It is straightforward to see that if ρ is a semi norm then its open and closed disks $B\rho(r-)$ and $B\rho(r-)$ are convex, balanced, and absorbing.

A semi-norm is determined by its unit disks. If $r_v = \|v\|_\rho > 0$ then

$$\|v/r\|_\rho \;\begin{cases} > 1 & \text{if } r < r_v \\ = 1 & \text{if } r = r_v \\ < 1 & \text{if } r > r_v. \end{cases}$$

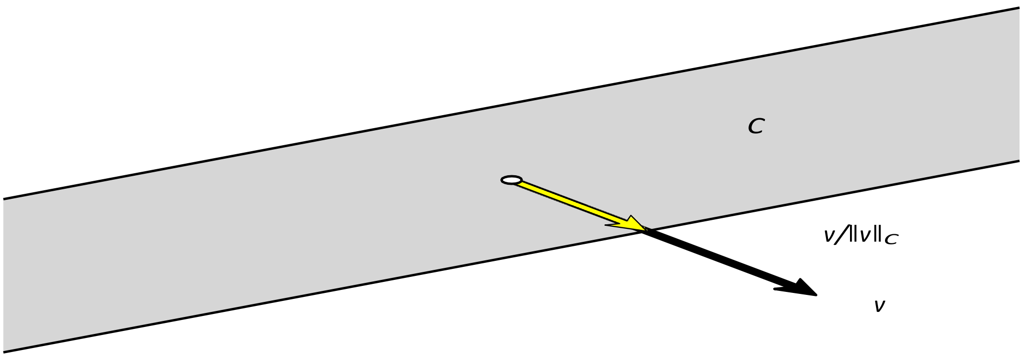
we have

$$\|v\|\rho = \inf\{r > 0 \,|\, v/r \in B\}$$

for B equal to either $B_\rho(1)$ or $B_\rho(1-)$.

Conversely, suppose C to be an absorbing subset of V. The intersection of the line $\mathbb{R}\cdot v$ with V is an interval, possibly infinite, around 0. Since C is absorbing, there exists $r > 0$ such that $v/r \in C$. Define

$$\|v\|_C = \inf\{\lambda \ge 0 \,|\, v/\lambda \in C\}$$



For example, if v lies on a line inside C then $\|v\|_C = 0$.

Proposition. If C is convex, balanced, and absorbing then $\rho(v) = \|v\|_C$ is a semi norm [8].

If C is not convex or balanced then it can be replaced by its convex, balanced hull, so the first two of these requirements are not onerous. But its not being absorbing is fatal—in that case, the semi norm will be infinite almost everywhere. So in practice it is that condition that one has to be careful about. Similarly, if one isgiven a formula for a semi norm, the important thing to check is that it be finite.

Proof. It is immediate that $\|cv\|_C = c\|v\|_C$ for $c > 0$, and since C is balanced it is immediate that $\|cv\|_C = \|v\|_C$ for $|c| = 1$. Finally, the function $\rho(v) = \|v\|_C$ is convex since C is.

The semi norm determined by C is called its **gauge** [**100**].

Several such sets C may determine the same semi norm. For example both the open and closed unit disks of ρ determine ρ. The correspondance becomes bijective if we impose a simple condition on C. I'll call a subsetof V **linearly open** if its intersection with any real line is open.

Lemma. If ρ is a convex real valued function then the region $\rho < c$ is linearly open.

For example, the open unit disk defined by a semi norm is linearly open.

Proof. Suppose P to be a point in V such that $\rho(P) < c$. I must show that every real line in V containing P contains also an open interval around P. It suffices for this to show that if Q is any other point in the vector space V, then points on the initial part of the segment from P to Q also satisfy $\rho < c$. If $\rho(Q) < c$ then the whole segment $[P, Q]$ lies in the region $\rho < c$. Otherwise say $\rho(Q) \geq c > \rho(P)$ or $\rho(Q) - \rho(P) > c - \rho(P) > 0$.

$$\rho\big((1-t)P + Q\big) \leq (1-t)\rho(P) + t_\rho(Q) = \rho(P) + t(\rho(Q) - \rho(P))$$

so if we choose t small enough so that

$$t\,\rho(Q) - \rho(P) < c - \rho(P)\ or\ t < \frac{c - \rho(P)}{\rho(Q) - \rho(P)}$$

then

$$\rho\big((1-t)P + tQ) < \rho(P) + (c - \rho(P)\big) = c.$$

Proposition. The map associating to ρ the open unit disk $B_\rho(1-)$ where $\|v\|_\rho < 1$ is a bijection between semi norms and subsets of V that are convex, balanced, and linearly open [9].

Proof. The previous result says that if ρ is a semi norm then $B_\rho(1-)$ is convex, balanced, and linearly open. It remains to show that if C is convex, balanced, and linearly open then it determines a semi norm for which it is the open unit disk. For the first claim, it suffices to point out that a convex, balanced, linearly opne set isabsorbing. That's because for any v the line through 0 and v must contain some open interval around 0 andinside C. This means that we can define in terms of C the semi norm $\rho = \rho_C$.

Why is C the open unit disk for ρ? It must be shown that a point v lies in C if and only if $\|v\|_C < 1$. Since $C$is convex, balanced, and linearly open, the set $\{c \in R \mid cv \in C\}$ is an open interval around 0 in R. Hence if v lies in C, there exists $(1 + \varepsilon)v \in C$ also, and therefore $\|v\|_C \leq 1/(1 + \varepsilon) < 1$.

Conversely, suppose $\|v\|_C = r < 1$. If $r = 0$, then cv lies in C for all c in R. Otherwise, v/r is on the boundary of C—$v/\eta \in C$ for $\eta > r$ but $v/\eta \notin C$ for $\eta < r$. Since $r < 1$, there then exists some $r_+ < 1$ such that $v/r_+ \in C$, and since C is convex and v lies between 0 and v/r_+ the vector v also lies in C.

In general, linearly open sets are a very weak substitute for open ones in a vector space, but convex ones aremuch better behaved. Linearly open sets will occur again in the discussion of the Hahn Banach Theorem, in which convex linearly open sets play a role. For now, I content myself with the following observation:

Proposition. In a finite dimensional vector space, every convex linearly open set is open [106-110].

Proof. Let U be a linearly open subset of V. We must show that for every point of U there exists some neighbourhood contained in U. We may as well assume that point to be the origin. Let (e_i) be a basis of V. There exists $c > 0$ such that all $\pm ce_i$ are in U, and since U is their convex hull, which is a neighbourhood of the origin.

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