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Study of Constructing a Compact Operator from Finite Rank Operators

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ABSTRACT
In this present paper, we studied about constructing a compact operator from finite rank operators. The purpose of this paper is to first review some concepts from Functional Analysis and Operator Algebra, then to apply these concepts to an in-depth introduction to Compact Operators and the Spectra of Compact Operators, leading to The Fredholm Alternative. [1-2].
Keywords: Topology, Compact Operators, Vector- Space, Hilbert Spaces, Homomorphic, Functional Analysis.

I. INTRODUCTION

A subset M of a topological space R is called compact, if every open covering of M contains a finite sub covering. A subset of a topological space R is called relatively compact if M is contained in a compact subset of R. A subset M of a topological vector space is called bounded if corresponding to every zero neighbourhood U there exists a $\alpha > 0$ such that $\alpha U \supset M$.[3-5].

Theorem: In a Topological Vector space

(a) Every subst of a bounded set is bounded.
(b) The continuous image of abounded set is bounded.
(c) The closed envelope of abounded set is bounded.
(d) Every compact set M is bounded.
(e) The union of finitely many and the intersection of arbitrary number of bounded sets is bounded.
(f) If M, N are bounded sets then the sets M+N and λM λ∈∅ are also bounded.

II. RESULTS AND DISCUSSION

Definition 1. Let X and Y *be* normed spaces. A linear transformation $T \in L(X, Y)$ is *compact* if, for any bounded sequence $\{x_n\}$ in X, the sequence Tx_n in Y contains a convergent subsequence. The set of compact transfor- mations in $L(X, Y)$ will be denoted by $K(X, Y)$.

Theorem 2. *Let X and Y be normed spaces an let* $T \in K(X, Y)$. *Then T is bounded. Thus* $K(X, Y) \subset B(X, Y)$.

Proof. Suppose that T is not bounded. Then for each integer $n \geq 1$ there exists a unit vector x_n such that $\| Tx_n \| \geq n$. Since the sequence $\{x_n\}$ is bounded, by the compactness of T there exists a subsequence $\{Tx_{n(r)}\}$ which converges. This contradicts $\| Tx_{n(r)} \| \geq n(r)$. (ie. convergence implies boundedness)[6].

Theorem 3. *Let X, Y, Z be normed spaces*

1. *If $S, T \in K(X, Y)$ and $\alpha, \beta \in \mathbb{C}$ then $\alpha S + \beta T$ is compact. Thus $K(X,Y)$ is a linear subspace of $B(X,Y)$.*
2. *If $S \in B(X, Y)$, $T \in B(Y, Z)$ and at least on of the operators S, T is compact, then $TS \in B(X, Z)$ is compact.*

Proof.1. Let $\{x_n\}$ be a bounded sequence in X. Since S *is* compact, there is a subsequence $\{x_{n(r)}\}$ such that $\{Sx_{n(r)}\}$ converges. Then, since $\{x_{n(r)}\}$ is bounded and T is compact, there is a subsequence $\{x_{n(r(s))}\}$ of the sequence $\{x_{n(r)}\}$ such that $\{Tx_{n(r(s))}\}$ converges. Since the sum of con- vergent sequences converges, it follows that the sequence $\{\alpha Sx_{n(r(s))}+\beta Tx_{n(r(s))}\}$ converges. Thus $\alpha S + \beta T$ is compact.

2. Let $\{x_n\}$ be a bounded sequence in X. If S is compact then there is a subsequence $\{x_{n(r)}\}$ such that $\{Sx_{n(r)}\}$ converges. Since T is bounded (and so is continuous), the sequence $\{TSx_{n(r)}\}$ converges. Thus TS is compact. If S is bounded but not compact the the sequence $\{Sx_n\}$ is bounded. Then since T must be compact, there is a subsequence $\{Sx_{n(r)}\}$ such that $\{TSx_{n(r)}\}$ converges, and again TS is compact.

For simplicity of notation we will make the following change (when it does not obscure the meaning of statements) $\{x_{n(r)}\}$, $\{x_{n(r(s))}\} \to \{x_n\}$.

Theorem 4. *Let X, Y be normed spaces and $T \in B(X, Y)$.*

1. *If T has finite rank then T is compact.*
2. *If either dim(X) or dim(Y) is finite then T is compact.*

Proof. 1. Since T has finite rank, the space $Z =$ Im T is a finite-dimensionalnormed space. Furthermore, for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ is bounded in Z, so by the Bolzano-Weierstrass theorem this sequence must contain a convergent subsequence. Hence T is compact.[7-9].

2. If dim X is finite then $r(T) \leq$ dim X, so $r(T)$ is finite, while if dim Y is finite then clearly the dimension of Im T Y must be finite. Thus, in either case the result follows from the previous part of this proof

Theorem 5. *If X is an infinite-dimensional normed space then the identity operator I on X is not compact.*

Proof. Since X is an infinite-dimensional normed space shows there exists a sequence of unit vectors $\{x_n\}$ in X which does not have any convergent subsequence. Hence the sequence $\{Ix_n\} = \{x_n\}$ cannot have a convergent subsequence, and so the operator I is not compact.

Corollary 6. *If X is an infinite-dimensional normed space and $T \in K(X)$ then T is not invertible.*

Proof. Suppose that T is invertible. Then, by Theorem 3, the identity oper- ator $I = T^{-1}T$ on X must be compact. Since X is infinite-dimensional this contradicts Theorem 5.

Theorem 7. *Let X, Y be normed spaces and let $T \in L(X, Y)$.*

1. *T is compact if and only if, for every bounded subset $A \subset X$, the set $T(A) \subset Y$ is relatively compact.*
2. *If is compact the $\overline{Im\ (T)}$ and Im(T) are separable.*

Proof. 1. Suppose that T is compact. Let A X be bounded and suppose that y_n is an arbitrary sequence in $\overline{T\ (A)}$. Then for each $n \in \mathbb{N}$, there exists

$x_n \in A$ such that $||y_n - Tx_n|| < n^{-1}$, and the sequence $\{x_n\}$ *is* bounded since A is bounded.

Thus, by compactness of T, the sequence $\{Tx_n\}$ contains a convergent subsequence, and hence $\{y_n\}$ contains a convergent subsequence with limit in $\overline{T(A)}$. Since $\{y_n\}$ is arbitrary, this shows that $\overline{T(A)}$ is compact. Now suppose that for every bounded subset $A \subset X$ the set $T(A) \subset Y$ is relatively compact. Then for any bounded sequence $\{x_n\}$ in X the sequence $\{Tx_n\}$ lies in a compact set, and hence contains a convergent subsequence. Thus T is compact.

2. For any $r \in$ N, let $R_r = T(B_r(0)) \subset Y$ be the image of the ball $B_r(0) \subset X$. Since T is compact, the set R_r is relatively compact and so is separable.Furthermore, since Im T equals the countable union $\cup_{r=1}^{\infty} \mathrm{R_r}$ it must also be separable. Finally, if a subset of Im T is dense in Im T then it is also dense in $\overline{\mathrm{Im}T}$(ie. by Definition $S \subset$ Im T we have $\bar{S} = \mathrm{Im}\,T$, and $\mathrm{ImT} \subset \overline{\mathrm{Im}T}$), so $\overline{\mathrm{Im}T}$ is separable.[10-12].

Theorem 8. *If X is a normed space, Y is a Banach space and $\{T_k\}$ is a sequence in K(X,Y) which converges to an operator $T \in B(X, Y)$, then T is compact. Thus K(X,Y) is closed in B(X,Y).*

Proof. Let $\{x_n\}$ be a bounded sequence in X. By compactness, there exists a subsequent of $\{x_n\}$, which we will label $x_{n(1,\mathrm{r})}$ (where r = 1,2,...), such that the sequence $T_1 x_{n(2,r)}$ converges. Similarly, there exists a subsequence $x_{n(2,\mathrm{r})}$ of $x_{n(1,\mathrm{r})}$ such that $T_2 x_{n(2,r)}$ coverage's. Also, $T_1 x_{n(2,r)}$ coverages since it is a subsequence of $T_1 x_{n(2,r)}$. Repeating this process inductively, we see that for each $j \in \mathbb{N}$ there is a subsequence $x_{n(\mathrm{j,r})}$ with the property: for any $k \le j$ the sequence $\{T_{k x_{n(j,r)}}\}$ converges. Letting n(r) = n(r,r), for $r \in \mathbb{N}$, we obtain a single sequence $\{x_{n(r)}\}$ with the property that, for each fixed $k \in \mathbb{N}$, the sequence$\{T_k x_{n(r)}\}$ converges as $r \to \infty$. This so-called "Cantor diagonalization" type argument is necessary to obtain a single sequence which works simultaneously for all the operators T_k, $k \in \mathbb{N}$.

We will now show that the sequence $\{Tx_{n(r)}\}$ converges. We do this by showing that $\{Tx_{n(r)}\}$ is a Cauchy sequence, and hence is convergent since Y is a Banach space.

Let $\epsilon > 0$ be given. Since the subsequence $\{x_{n(r)}\}$ is bounded there exists $M > 0$ such that $\| x_{n(r)} \| \le M$, for all $r \in \mathbb{N}$. Also, since $\| T_k - T \| \to 0$, as $k \to \infty$, there exists an integer $K \ge 1$ such that $\| T_K - T \| < \frac{\epsilon}{3M}$. Next, since $\{T_K x_{n(r)}\}$ converges there exists an integer $R \ge 1$ such that if $r, s \ge R$ then $\| T_K x_{n(r)} - T_K x_{n(s)} \| > \frac{\epsilon}{3}$

Now we have, for $r, s \ge R$

$$\begin{aligned}
&\|Tx_{n(r)}\| \\
&< \|Tx_{n(r)} - T_K x_{n(r)}\| + \|T_K x_{n(r)} - T_K x_{n(s)}\| \\
&\qquad + \|T_K x_{n(s)} - Tx_{n(s)}\| \\
&\qquad \le \|T_K - T\|\|x_{n(s)}\| \\
&\qquad < \frac{\epsilon}{3M} M + \frac{\epsilon}{3} + \frac{\epsilon}{3M} M = \epsilon
\end{aligned}$$

which proves that $\{Tx_{n(r)}\}$ is a Cauchy sequence.

Corollary 9. *If X is a normed space, Y is a Banach space and T_k is a sequence of bounded, finite rank operators which converge to $T \in B(X, Y)$, then T is compact.*

Proof. Theorem 4.2 shows $\{T_k\}$ are compact then we apply Theorem 8.

Theorem 10. *If X is a normed space,* H *is a Hilbert space and $T \in K(X, \mathrm{H})$, then there is a sequence of finite rank operators $\{T_k\}$ which converges to T in B(X,*H*).*

Proof. If T itself had finite rank the result would be trivial, so we consider the case that it does not. By Theorem 7 the set Im T is an infinite-dimensional, separable Hilbert space, so it has an orthonormal basis $\{e_n\}$. For each integer $k \ge 1$, let P_k be the orthogonal projection from $\overline{Im\,T}$ onto the linear subspace $\mathrm{M}_k = \mathrm{Sp}\{e_1, \ldots, e_k\}$, and let $T_k = P_k T$. Since Im $T_k \subset \mathrm{M}_k$, the operator T_k has finite rank. We will show that $\|T_K - T\| \to 0$ *as* $k \to \infty$.

Suppose that this is not true. Then, after taking a subsequence of the sequence $\{T_k\}$ if necessary, there is an $\epsilon > 0$ such that $\| T_k - T \| \geq \epsilon$ for all k. Thus there exists a sequence of unit vectors $x_k \in X$ such that $\| (T_k - T) x_k \| \geq \frac{\epsilon}{2}$ for all k. Since T is compact, we may suppose that $Tx_k \to y$, for some $y \in$ H. (after again taking a subsequence, if necessary). Now, using the representation of P_m in Corollary 5.8, we have,

$$(T_k - T)x_k = (P_k - I)Tx_k$$
$$= (P_k - I)y + (P_k - I)(Tx_k - y)$$
$$= -\sum_{n=k+1}^{\infty} (y, e_n) + (P_k - I)\|Tx_K - y\|$$

Hence, by taking the norms and using $P_k = 1_{\|}$ we deduce(using properties of norms) that

$$\frac{\epsilon}{2} \leq \|(T_K - T)x_k\|$$
$$\leq \left(\sum_{n=k+1}^{\infty} (y, e_n)^2)^{\frac{1}{2}} + 2\|Tx_K - y\| \right)$$

The right-hand side of this inequality tends to zero as k which is a contradiction, and so proves the theorem.

Lemma 11 *If $\mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$, then r(T) = r(T*) (either as finite numbers or as ∞). In particular, T has finite rank if and only if T* has finite rank.*

Proof. Suppose first that r(T) < ∞. For any$x \in \mathcal{H}$, we write that the orthogonal decomposition of x with respect to Ker T* as $x = u + v$, with $u \in Ker\, T^*$ and $v \in (KerT^*)^{\perp} = \overline{ImT} = ImT$ (since r(T) < ∞ Thus $T^*x = T^*(u + v) = T^*v$, and hence IM $T^* = T^*(ImT)$, which implies that r(T^*) $\leq r(T)$. Thus, r(T^*) $\leq r(T) < \infty$.

Applying this result to T^*, and using (T^*)*=T, we also see that r(T) ≤ r(T^*) when r(T^*) $< \infty$. This proves the lemma when both the ranks are finite, and also shows that it is impossible for one rank to be finite and the other infinite, and so also proves the infinite rank case.[13].

Theorem 12. *If$\mathcal{H}$ is a Hilbert space and$T \in B(\mathcal{H})$, then T is compact if and only if$T^*{}_{\overline{2}}$is compact.*

Proof : Suppose that T is compact. Then by Theorem 6.10 there is a sequence of finite rank operation $\{ T_n \}$, such that $\|T_n - T\| \to 0$. By Lemma 6.11, each operator T_n^* has finite rank and, by Theorem 5.4 $\|T_n^* - T^*\| = \|T_n - T\| \to 0$. Hence it follows from Corollary 9 that T* is compact. Thus, if T is compact then T* is compact. It now follows from this result and (T*)* = T that if T* is compact the T is compact, which completes the proof.

III.CONCLUSION

Constructing a Compact Operator from Finite Rank Operators.

One can build the Compact Operator $T \in B\,(\ell^2)$ defined by $T\{a_n\}\{n^{-1}a_n\}$ from the finite rank operators $T_k \in B(\ell^2)$ defined by $T\{a_n\}\{n^{-1}a_n\}$ from the finite rank operators $T_k \in B(\ell^2)$ where $T\{a_n\} = \{b_n^k\}$ and $b_n^k = n^{-1}a_n$ when $n \leq k$ and $b_n^k = 0$ when $n > k$. Thus for any $a \in \ell^2$

$$\|(T_k - T)a\|^2$$
$$= \sum_{n=k+1}^{\infty} \frac{|a_n|^2}{n^2}$$
$$\leq (k + 1)^{-2} \sum_{n=k+1}^{\infty} |a_n|^2 \leq \frac{\|a\|^2}{(k+1)^{-2}}$$

and therefore

$$\|T_k - T\| \leq (k+1)^{-1} \to 0$$

The result follows from Corollary 9

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